ON SEMIGROUPS OF OPERATORS IN LOCALLY CONVEX SPACES

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1. The following problem is investigated in this note: Let $\{T(\xi)\}_{\xi>0}$ be a semigroup of continuous linear operators in a locally convex space E. Under which conditions (e.g. measurability conditions) is it true that the functions $\xi\rightarrow T(\xi)x$ are continuous for every $x\in E$? The point of departure is the paper [6] by Singbal-Vedak.

In [6], the following generalization of earlier results for Banach spaces by Dunford [1] and Hille [2] was proved: If $\{T(\xi)\}_{\xi>0}$ is a measurable semigroup such that $\{T(\xi)\}_{\alpha\leq\xi\leq\beta}$ is equicontinuous for any $[\alpha, \beta]\subset(0, \infty)$, then the functions $\xi\rightarrow T(\xi)x$ are continuous. It is shown in Proposition 1 that a weaker equicontinuity condition, which is only required to hold locally at the origin 0, implies the conclusion. Proposition 2 treats the special case of semigroups in LF spaces. Another theorem by Singbal-Vedak [6], generalizing a result of Phillips [4] for Banach spaces, says that if E is a Fréchet space and the semigroup is measurable, then $\xi\rightarrow T(\xi)x$ is continuous for every $x\in E$. In Proposition 3 it is shown that the conclusion remains true when E is merely a metrizable locally convex space.

2. The notation and definitions used in this note agree with [6]. We say that a semigroup of operators in a Hausdorff locally convex space E is weakly measurable (or, almost separably valued) if, for each $x\in E$, the function $\xi\rightarrow T(\xi)x$ is weakly measurable (or, almost separably valued). Observe that a measurable function $f:\mathbf{R}\rightarrow E$ is weakly measurable and almost separably valued, while the converse is known to hold if E is metrizable (by a generalization of a result of Pettis [3]). It seems to be an open problem whether the converse holds for nonmetrizable E, or not.

In Proposition 1, $\tau(E, E')$ denotes the Mackey topology on E (see e.g. [5]). Suppose $f:I\rightarrow E$ is a weakly measurable and almost separably valued function, where $I\subset\mathbf{R}$ is a compact interval, and suppose q is a $\tau(E, E')$-continuous seminorm on E. Then (i) $\xi\rightarrow q(f(\xi))$ is a measurable function, and (ii) for any $\epsilon>0$, there exists a continuous function $g:I\rightarrow E$ such that $\int_I q(f(\xi)-g(\xi))d\xi<\epsilon$ (these facts are, with small modifications, contained in [6, Proposition 1 and the proof of Proposition 2]).

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PROPOSITION 1. *Let* $\{T(\xi)\}_{\xi>0}$ *be a weakly measurable and almost separably valued semigroup of continuous linear operators in a locally convex space* E, *satisfying*:

(I) *for each neighborhood* U *of* 0 *and each continuous seminorm* p *on* E, *there exists a measurable set* $A\subset U\cap(0, \infty)$ *with* $m(A)>0$ *such that* $\{p\circ T(\xi)\}_{\xi\in A}$ *is an equicontinuous set of mappings of* $(E, \tau(E, E'))$ *into* $\boldsymbol{R}$.

Then $\xi\to T(\xi)x$ *is continuous for every* $x\in E$.

PROOF. Let $x\in E$ and $\xi>0$. Given any continuous seminorm p on E, we have to prove that $\lim_{\delta\to 0} p(T(\xi+\delta)x-T(\xi)x)=0$. Defining the point $\alpha=\xi/3$, condition (I) implies the existence of a measurable set $A\subset(0, \alpha)$ with $m(A)>0$ and a $\tau(E, E')$-continuous seminorm q on E such that $p(T(\eta)y)\leqq q(y)$ for $\eta\in A$ and $y\in E$. Given $\epsilon>0$, choose a continuous function g: $[-\alpha, 2\alpha]\to E$ such that

$$\int_{-\alpha}^{2\alpha} q(T(\xi-\eta)x-g(\eta))d\eta<\epsilon$$

(this is possible according to (ii) of the remark above). For any δ with $|\delta|<\alpha$ and any $\eta\in A$, we have

$$\begin{aligned} p(T(\xi+\delta)x-T(\xi)x) &= p(T(\eta)(T(\xi+\delta-\eta)x-T(\xi-\eta)x)) \\ &\leqq q(T(\xi+\delta-\eta)x-T(\xi-\eta)x). \end{aligned}$$

Since the function $\eta\to q(T(\xi+\delta-\eta)x-T(\xi-\eta)x)$ is measurable according to (i) above, integration with respect to η yields

$$\begin{aligned} m(A)p(T(\xi+\delta)x-T(\xi)x) &\leqq \int_A q(T(\xi+\delta-\eta)x-T(\xi-\eta)x)d\eta \\ &\leqq \int_0^\alpha q(T(\xi+\delta-\eta)x-g(\eta+\delta))d\eta \\ &\quad+\int_0^\alpha q(g(\eta+\delta)-g(\eta))d\eta \\ &\quad+\int_0^\alpha q(g(\eta)-T(\xi-\eta)x)d\eta<3\epsilon, \end{aligned}$$

for sufficiently small values of δ. Therefore the function $\eta\to T(\eta)x$ is continuous at the point ξ.

The following condition clearly implies that (I) of Proposition 1 holds: for each $[\alpha, \beta]\subset(0, \infty)$, $\{T(\xi)\}_{\alpha\leqq\xi\leqq\beta}$ is equicontinuous (cf. [6, Proposition 2]). In (I), however, A is allowed to depend on the seminorm p. The equicontinuity with respect to the $\tau(E, E')$-topology is in general a weaker condition than equicontinuity with respect to the given topology on E (suppose, for example, that we consider E with the weak topology $\sigma(E, E')$). Finally, (I) is a local condition at 0.

In the next proposition, we consider an LF space, i.e. a strict inductive limit of an increasing sequence of Fréchet spaces (see [**5**]; the space $\mathfrak{D}$ of distribution theory is an LF space). It is known that an LF space has the $\tau(E, E')$-topology, and it is barrelled and non-metrizable.

PROPOSITION 2. *Let E be an LF space with respect to the subspaces (E_n), and let $\{T(\xi)\}_{\xi>0}$ be a weakly measurable and almost separably valued semigroup of continuous linear operators in E. Then $\xi\to T(\xi)x$ is continuous for every $x\in E$ if, and only if, for every $x\in E$ and every $[\alpha, \beta]\subset(0, \infty)$, there exists an $n\in \boldsymbol{N}$ such that $\{T(\xi)x\}_{\alpha\leq\xi\leq\beta}\subset E_n$.*

PROOF. Assume $\xi\to T(\xi)x$ to be continuous. Then $\{T(\xi)x\}_{\alpha\leq\xi\leq\beta}$ is compact, and hence bounded in E, so it is contained in some E_n [**5**, p. 59]. For the converse, suppose p is a continuous seminorm on E and U is a neighborhood of 0. Choose an interval $[\alpha, \beta]\subset U\cap(0, \infty)$. In order to verify that (I) of Proposition 1 holds, it is sufficient to show that $\{p\circ T(\xi)\}_{\alpha\leq\xi\leq\beta}$ is pointwise bounded. For then the uniform boundedness principle for continuous seminorms, in the barrelled space E, implies that $\{p\circ T(\xi)\}_{\alpha\leq\xi\leq\beta}$ is equicontinuous.

Suppose there exists an $x\in E$ such that $\{p(T(\xi)x)\}_{\alpha\leq\xi\leq\beta}$ is unbounded. Then we find a sequence $(\xi_j)\subset[\alpha, \beta]$ and a point $\gamma\in[\alpha, \beta]$ with $\lim_{j\to\infty}\xi_j=\gamma$ and $p(T(\xi_j)x)\geqq j$ for all $j\in\boldsymbol{N}$. By assumption, there is an n such that $\{T(\xi)x\}_{\alpha/2\leq\xi\leq\beta}\subset E_n$. Let (p_k)be a sequence of continuous seminorms on E_n which defines the topology of E_n. For each k, there exists a continuous seminorm p_k' on E such that $p_k'|_{E_n}=p_k$. Since $\xi\to p_k'(T(\xi)x)$ of $(0, \infty)\to\boldsymbol{R}$ is measurable, the restriction of the function $\xi\to p_k(T(\xi)x)$ to the interval $[\alpha/2, \gamma]$ is measurable. From here on, proceed as in the proof of Proposition 3 of [**6**]. We find a set $F\subset[\alpha/2, \gamma]$ such that $\{T(\xi)x\}_{\xi\in F}$ is a bounded subset of E_n, and a point $\sigma_0\in(0, \alpha/2)$ satisfying the property: for each $j\in\boldsymbol{N}$, there exists $\eta_j\in F$ with $\sigma_0=\xi_j-\eta_j$. Then $p(T(\sigma_0)T(\eta_j)x)=p(T(\sigma_0+\eta_j)x)=p(T(\xi_j)x)\geqq j$ for each j, contradicting the fact that $T(\sigma_0)[\{T(\xi)x\}_{\xi\in F}]$ is bounded. We conclude that $\{p\circ T(\xi)\}_{\alpha\leq\xi\leq\beta}$ is pointwise bounded. By Proposition 1, $\xi\to T(\xi)x$ is continuous for every $x\in E$.

PROPOSITION 3. *Let $\{T(\xi)\}_{\xi>0}$ be a measurable semigroup of continuous linear operators in a metrizable locally convex space E. Then*

$\{T(\xi)\}_{\alpha\leq\xi\leq\beta}$ is equicontinuous for any $[\alpha, \beta]\subset(0, \infty)$, and $\xi\to T(\xi)x$ is continuous for every $x\in E$.

Proof. Let $\hat{E}$ denote the completion of E and, for each $\xi>0$, let $\hat{T}(\xi):\hat{E}\to\hat{E}$ be the continuous extension of $T(\xi)$. Then $\{\hat{T}(\xi)\}_{\xi>0}$ is a semigroup of continuous linear operators in the Fréchet space $\hat{E}$. Given any $\hat{x}\in\hat{E}$, choose a sequence (x_n) in E such that $\lim_{n\to\infty} x_n=\hat{x}$. The functions $\xi\to\hat{T}(\xi)x_n=T(\xi)x_n$ are measurable by assumption, and for each $\xi>0$ we have $\lim_{n\to\infty}\hat{T}(\xi)x_n=\hat{T}(\xi)\hat{x}$. Therefore $\xi\to\hat{T}(\xi)\hat{x}$ is measurable, i.e. $\{\hat{T}(\xi)\}_{\xi>0}$ is a measurable semigroup. Now Proposition 3 of [6] implies that, for any $[\alpha, \beta]\subset(0, \infty)$, $\{\hat{T}(\xi)\}_{\alpha\leq\xi\leq\beta}$ is equicontinuous, so $\{T(\xi)\}_{\alpha\leq\xi\leq\beta}$ is an equicontinuous set of mappings from E into E. Hence, (I) of Proposition 1 is satisfied, so $\xi\to T(\xi)x$ is continuous for every $x\in E$.

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