ON SEMIGROUPS OF OPERATORS IN LOCALLY
CONVEX SPACES

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1. The following problem is investigated in this note: Let
   \( \{ T(\xi) \}_{\xi > 0} \) be a semigroup of continuous linear operators in a locally
   convex space \( E \). Under which conditions (e.g. measurability condi-
   tions) is it true that the functions \( \xi \mapsto T(\xi)x \) are continuous for every
   \( x \in E \)? The point of departure is the paper [6] by Singbal-Vedak.

   In [6], the following generalization of earlier results for Banach
   spaces by Dunford [1] and Hille [2] was proved: If \( \{ T(\xi) \}_{\xi > 0} \) is a
   measurable semigroup such that \( \{ T(\xi) \}_{\alpha \leq \xi \leq \beta} \) is equicontinuous for
   any \( [\alpha, \beta] \subset (0, \infty) \), then the functions \( \xi \mapsto T(\xi)x \) are continuous. It
   is shown in Proposition 1 that a weaker equicontinuity condition,
   which is only required to hold locally at the origin \( 0 \), implies the
   conclusion. Proposition 2 treats the special case of semigroups in
   LF spaces. Another theorem by Singbal-Vedak [6], generalizing a
   result of Phillips [4] for Banach spaces, says that if \( E \) is a Fréchet
   space and the semigroup is measurable, then \( \xi \mapsto T(\xi)x \) is continuous
   for every \( x \in E \). In Proposition 3 it is shown that the conclusion
   remains true when \( E \) is merely a metrizable locally convex space.

2. The notation and definitions used in this note agree with
   [6]. We say that a semigroup of operators in a Hausdorff locally
   convex space \( E \) is weakly measurable (or, almost separably valued)
   if, for each \( x \in E \), the function \( \xi \mapsto T(\xi)x \) is weakly measurable (or, almost separably valued). Observe that a measurable function
   \( f: R \to E \) is weakly measurable and almost separably valued, while
   the converse is known to hold if \( E \) is metrizable (by a generalization
   of a result of Pettis [3]). It seems to be an open problem whether
   the converse holds for nonmetrizable \( E \), or not.

   In Proposition 1, \( \tau(E, E') \) denotes the Mackey topology on \( E \)
   (see e.g. [5]). Suppose \( f: I \to E \) is a weakly measurable and almost
   separably valued function, where \( I \subset R \) is a compact interval, and
   suppose \( q \) is a \( \tau(E, E') \)-continuous seminorm on \( E \). Then (i) \( \xi \mapsto q(f(\xi)) \)
   is a measurable function, and (ii) for any \( \epsilon > 0 \), there exists a con-
   tinuous function \( g: I \to E \) such that \( \int_I q(f(\xi) - g(\xi))d\xi < \epsilon \) (these facts
   are, with small modifications, contained in [6, Proposition 1 and the
   proof of Proposition 2]).

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Proposition 1. Let \( \{ T(\xi) \}_{\xi > 0} \) be a weakly measurable and almost separably valued semigroup of continuous linear operators in a locally convex space \( E \), satisfying:

(I) for each neighborhood \( U \) of 0 and each continuous seminorm \( p \) on \( E \), there exists a measurable set \( A \subseteq U \cap (0, \infty) \) with \( m(A) > 0 \) such that \( \{ p \circ T(\xi) \}_{\xi \in A} \) is an equicontinuous set of mappings of \( (E, \tau(E, E')) \) into \( \mathbb{R} \).

Then \( \xi \to T(\xi)x \) is continuous for every \( x \in E \).

Proof. Let \( x \in E \) and \( \xi > 0 \). Given any continuous seminorm \( p \) on \( E \), we have to prove that \( \lim_{\xi \to 0} p(T(\xi + \delta)x - T(\xi)x) = 0 \). Defining the point \( \alpha = \xi/3 \), condition (I) implies the existence of a measurable set \( A \subseteq (0, \alpha) \) with \( m(A) > 0 \) and a \( \tau(E, E') \)-continuous seminorm \( q \) on \( E \) such that \( p(T(\eta)y) \leq q(y) \) for \( \eta \in A \) and \( y \in E \). Given \( \varepsilon > 0 \), choose a continuous function \( g: [-\alpha, 2\alpha] \to E \) such that

\[
\int_{-\alpha}^{2\alpha} q(T(\xi - \eta)x - g(\eta))d\eta < \varepsilon
\]

(this is possible according to (ii) of the remark above). For any \( \delta \) with \( |\delta| < \alpha \) and any \( \eta \in A \), we have

\[
p(T(\xi + \delta)x - T(\xi)x) = p(T(\eta)(T(\xi + \delta - \eta)x - T(\xi - \eta)x)) \leq q(T(\xi + \delta - \eta)x - T(\xi - \eta)x).
\]

Since the function \( \eta \to q(T(\xi + \delta - \eta)x - T(\xi - \eta)x) \) is measurable according to (i) above, integration with respect to \( \eta \) yields

\[
m(A)p(T(\xi + \delta)x - T(\xi)x) \leq \int_A q(T(\xi + \delta - \eta)x - T(\xi - \eta)x)d\eta
\]

\[
\leq \int_0^\alpha q(T(\xi + \delta - \eta)x - g(\eta + \delta))d\eta
\]

\[
+ \int_0^\alpha q(g(\eta + \delta) - g(\eta))d\eta
\]

\[
+ \int_0^\alpha q(g(\eta) - T(\xi - \eta)x)d\eta < 3\varepsilon,
\]

for sufficiently small values of \( \delta \). Therefore the function \( \eta \to T(\eta)x \) is continuous at the point \( \xi \).

The following condition clearly implies that (I) of Proposition 1 holds: for each \( [\alpha, \beta] \subseteq (0, \infty) \), \( \{ T(\xi) \}_{\xi \in \alpha \leq \xi \leq \beta} \) is equicontinuous (cf. [6, Proposition 2]). In (I), however, \( A \) is allowed to depend on the
seminorm \( p \). The equicontinuity with respect to the \( \tau(E, E') \)-topology is in general a weaker condition than equicontinuity with respect to the given topology on \( E \) (suppose, for example, that we consider \( E \) with the weak topology \( \sigma(E, E') \)). Finally, (1) is a local condition at 0.

In the next proposition, we consider an LF space, i.e. a strict inductive limit of an increasing sequence of Fréchet spaces (see [5]; the space \( \mathcal{D} \) of distribution theory is an LF space). It is known that an LF space has the \( \tau(E, E') \)-topology, and it is barrelled and non-metrizable.

**Proposition 2.** Let \( E \) be an LF space with respect to the subspaces \( (E_n) \), and let \( \{ T(\xi) \}_{\xi>0} \) be a weakly measurable and almost separably valued semigroup of continuous linear operators in \( E \). Then \( \xi \to T(\xi)x \) is continuous for every \( x \in E \) if, and only if, for every \( x \in E \) and every \( [\alpha, \beta] \subset (0, \infty) \), there exists an \( n \in \mathbb{N} \) such that \( \{ T(\xi)x \}_{\alpha \leq \xi \leq \beta} \subseteq E_n \).

**Proof.** Assume \( \xi \to T(\xi)x \) to be continuous. Then \( \{ T(\xi)x \}_{\alpha \leq \xi \leq \beta} \) is compact, and hence bounded in \( E \), so it is contained in some \( E_n \) [5, p. 59]. For the converse, suppose \( p \) is a continuous seminorm on \( E \) and \( U \) is a neighborhood of 0. Choose an interval \( [\alpha, \beta] \subset U \cap (0, \infty) \).

In order to verify that (1) of Proposition 1 holds, it is sufficient to show that \( \{ p(T(\xi)x) \}_{\alpha \leq \xi \leq \beta} \) is pointwise bounded. For then the uniform boundedness principle for continuous seminorms, in the barrelled space \( E \), implies that \( \{ p(T(\xi)x) \}_{\alpha \leq \xi \leq \beta} \) is equicontinuous.

Suppose there exists an \( x \in E \) such that \( \{ p(T(\xi)x) \}_{\alpha \leq \xi \leq \beta} \) is unbounded. Then we find a sequence \( \{ \xi_j \} \subset [\alpha, \beta] \) and a point \( \gamma \in [\alpha, \beta] \) with \( \lim_{j \to \infty} \xi_j = \gamma \) and \( p(T(\xi_j)x) \geq j \) for all \( j \in \mathbb{N} \). By assumption, there is an \( n \) such that \( \{ T(\xi)x \}_{\alpha \leq \xi \leq \beta} \subseteq E_n \). Let \( (p_k) \) be a sequence of continuous seminorms on \( E_n \) which defines the topology of \( E_n \). For each \( k \), there exists a continuous seminorm \( p'_k \) on \( E \) such that \( p'_k \big| E_n = p_k \). Since \( \xi \to p'_k(T(\xi)x) \) of \( (0, \infty) \to \mathbb{R} \) is measurable, the restriction of the function \( \xi \to p_k(T(\xi)x) \) to the interval \( [\alpha/2, \gamma] \) is measurable. From here on, proceed as in the proof of Proposition 3 of [6]. We find a set \( F \subset [\alpha/2, \gamma] \) such that \( \{ T(\xi)x \}_{\xi \in F} \) is a bounded subset of \( E_n \), and a point \( \sigma_0 \in (0, \alpha/2) \) satisfying the property: for each \( j \in \mathbb{N} \), there exists \( \eta_j \in F \) with \( \sigma_0 = \xi_j - \eta_j \). Then \( p(T(\sigma_0 + \eta_j)x) = p(T(\sigma_0 + \eta_j)x) \leq j \) for each \( j \), contradicting the fact that \( T(\sigma_0) \{ T(\xi)x \}_{\xi \in F} \) is bounded. We conclude that \( \{ p(T(\xi)x) \}_{\alpha \leq \xi \leq \beta} \) is pointwise bounded.

**Proposition 3.** Let \( \{ T(\xi) \}_{\xi \geq 0} \) be a measurable semigroup of continuous linear operators in a metrizable locally convex space \( E \). Then
\{ T(\xi) \}_{\alpha \leq \xi \leq \beta} is equicontinuous for any \([\alpha, \beta] \subseteq (0, \infty)\), and \( \xi \to T(\xi)x \) is continuous for every \( x \in E \).

**Proof.** Let \( \hat{E} \) denote the completion of \( E \) and, for each \( \xi > 0 \), let \( \hat{T}(\xi) : \hat{E} \to \hat{E} \) be the continuous extension of \( T(\xi) \). Then \( \{ \hat{T}(\xi) \}_{\xi > 0} \) is a semigroup of continuous linear operators in the Fréchet space \( \hat{E} \).

Given any \( \hat{x} \in \hat{E} \), choose a sequence \( (x_n) \) in \( E \) such that \( \lim_{n \to \infty} x_n = \hat{x} \). The functions \( \xi \to T(\xi)x_n = T(\xi)x_n \) are measurable by assumption, and for each \( \xi > 0 \) we have \( \lim_{n \to \infty} T(\xi)x_n = T(\xi)\hat{x} \). Therefore \( \xi \to T(\xi)\hat{x} \) is measurable, i.e. \( \{ T(\xi) \}_{\xi > 0} \) is a measurable semigroup. Now Proposition 3 of [6] implies that, for any \( [\alpha, \beta] \subseteq (0, \infty) \), \( \{ T(\xi) \}_{\alpha \leq \xi \leq \beta} \) is equicontinuous, so \( \{ T(\xi) \}_{\alpha \leq \xi \leq \beta} \) is an equicontinuous set of mappings from \( E \) into \( E \). Hence, (1) of Proposition 1 is satisfied, so \( \xi \to T(\xi)x \) is continuous for every \( x \in E \).

**References**


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