A METRIZATION THEOREM FOR LINEARLY ORDERABLE SPACES

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A topological space $X$ is linearly orderable if there is a linear ordering of the set $X$ whose open interval topology coincides with the topology of $X$. It is known that if a linearly orderable space is semi-metrizable then it is, in fact, metrizable [1]. We will use this fact to give a particularly simple metrization theorem for linearly orderable spaces, namely that a linearly orderable space is metrizable if and only if it has a $G_δ$ diagonal. This is an interesting analogue of the well-known metrization theorem which states that a compact Hausdorff space is metrizable if it has a $G_δ$ diagonal.

**Theorem.** A linearly orderable space with a $G_δ$ diagonal is metrizable.

**Proof.** Suppose that $X$ is linearly orderable and that the set $\Delta = \{(x, x) \mid x \in X\}$ is a $G_δ$ in the space $X \times X$, say $\Delta = \bigcap_{n=1}^{\infty} W(n)$. We may assume that $W(n+1) \subseteq W(n)$ for each $n \geq 1$. For each $x \in X$ and each $n \geq 1$, there is an open interval $g(n, x)$ in $X$ such that $(x, x) \in g(n, x)$ and such that $g(n+1, x) \subseteq g(n, x)$. It is easily verified that the collection $\{g(n, x) \mid n \geq 1\}$ is a local base at $x$ for each $x \in X$.

Suppose that $y \in X$ and that $<x(n)>$ is a sequence in $X$ such that $y \in g(n, x(n))$ for each $n \geq 1$. Clearly, if $z \in \bigcap_{n=1}^{\infty} g(n, x(n))$ then $(z, y) \in \Delta$. Therefore, if $r < y < s$, we can find an integer $N$ such that $r \in g(n, x(n))$ whenever $n \geq N$. Therefore, $x(n) \in [r, s]$ for $n \geq N$. Hence $<x(n)>$ converges to $y$. It now follows from [2, Theorem 3.2] that $X$ is semimetrizable. Therefore, $X$ is metrizable [1, Theorem 4.25].

**Remark 1.** Note that our theorem cannot be proved by applying the metrization theorem for compact Hausdorff spaces mentioned above to the space $X^+$, the natural order compactification of $X$, since it may happen that $X$ has a $G_δ$ diagonal while $X^+$ does not. Consider, for example, the “interior points” $\{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 < y < 1\}$ of the unit square with the lexicographic order topology. This space has a $G_δ$ diagonal and is metrizable, but since it is not separable, its order compactification cannot have a $G_δ$ diagonal.

**Remark 2.** It should be pointed out that a linearly orderable space $X$ in which closed sets are $G_δ$'s need not be metrizable, even

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if $X$ is compact. The space $[0, 1] \times \{0, 1\}$ with the lexicographic order topology provides an example.

**Remark 3.** As an application of our metrization theorem, we observe that the Sorgenfrey line (i.e., the real line with the left half open interval topology) is not linearly orderable since it is non-metrizable and has a $G_\delta$ diagonal.

**Bibliography**


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