A NOTE ON PUNCTURED DISKS IN A 2-MANIFOLD[1]

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Let D_n be a punctured disk with $n-1$ holes and let M be a separable 2-manifold. Let $f: D_n \to M$ be a map which is an embedding of the boundary ∂D_n of D_n.

THEOREM. *Some subfamily of the family*

$$C_1 \cup \cdots \cup C_n = f(\partial D_n)$$

of simple loops contains C_1 *and bounds an embedded punctured disk in* M.

This was proved in [1] for $n=1$, and in [2] for $n=1, 2$ when M is closed and orientable.

PROOF. We assume $n>1$. Observe that $C_1 \cup \cdots \cup C_n$ is a boundary in the mod 2 homology of M. Thus any C_i is homologous to $\bigcup_{j\neq i} C_j$. If C_i were a generating loop of a handle or a Mobius band in M not meeting any other C_j, then it would generate a direct summand of the first homology group with $\bigcup_{j\neq i} C_j$ in the complementary summand. In particular each C_i is two sided.

Form a manifold N by cutting M apart along the curves C_i, $i=2, \cdots, n$ (but not C_1) and attaching disks E_i, E_i' to the resulting boundary components. Form a space $\tilde{N}$ by identifying E_i with E_i' for $i=2, \cdots, n$. Then C_1 is null homotopic in $\tilde{N}$ since $C_2, \cdots, C_n$ bound disks. We show that C_1 is null homotopic in N.

Suppose not and consider what happens with the fundamental groups as we paste together one pair of disks at a time. If we bring together two components of N then we get the free product of their fundamental groups. If we paste together disks on the same component then we add a new generator and no relations. In neither case does C_1 become null homotopic.

By [1], C_1 bounds a disk D in N. Remove from D any of the E's which it contains and take the closure of the resulting subset of M. If at most one of each pair E_i, E_i' were contained in D, then the closure would be the desired punctured disk. If some pair E_i, E_i' were contained in D, then the closure would include a handle of M having C_i as a generating loop and not meeting any other C_j. This contradiction finishes the proof.

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