THE COEFFICIENTS OF STARLIKE FUNCTIONS

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1. In his thesis [1] Bombieri shows that there exist constants c_n such that if $f(z)=z+a_2z^2+\cdots$ is univalent in $|z|<1$ then

$$(1)\qquad |\operatorname{Re}(n-a_n)| \leqq c_n \operatorname{Re}(2-a_2).$$

The exact size of these coefficients is unknown. In his problem book [4, Problem 6.3] Hayman asked if there exist constants d_n such that

$$(2)\qquad |n-|a_n|| \leqq d_n(2-|a_2|)$$

for any f in the class of normalized univalent functions in the unit disc.

The proof of the local maximum theory for the coefficients of univalent functions by Bombieri [2] and by Garabedian and Schiffer [3] gives strength to the conjecture that these d_n exist, but no estimate of their size is available from these papers. However, we can make a precise estimate of the size of these coefficients for the subclass of the univalent functions consisting of the starlike functions. In this paper, we show that if $f(z)$ is starlike, then

$$(3)\qquad n-a_n=\gamma_n(2-a_2)$$

where the γ_n are constants depending on the function f but satisfying

$$(4)\qquad |\gamma_n| \leqq n(n-1)(n+1)/6$$

for any starlike function. Furthermore, no better estimate than (4) is possible.

2. We denote by S^* the class of starlike functions, that is functions which are regular in $U=\{z \mid |z|<1\}$, are normalized to have the series expansion $f(z)=z+a_2z^2+a_3z^3+\cdots$ and satisfy

$$\operatorname{Re}\{zf'(z)/f(z)\}>0$$

in U [7]. We denote by K the class of close-to-convex functions [6], [8]. These are the functions which are regular in U and which satisfy

$$(5)\qquad \operatorname{Re}\{zg'(z)/f(z)\}>0$$

in U for some function $f\in S^*$. In this case we say that g is close-to-

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convex with respect to the starlike function f. If

$$f(z) = b_0 + b_1 z + b_2 z^2 + \cdots$$

is in the class K, then Reade [8] has shown that

$$|b_n| \leqq n|b_1|. \tag{6}$$

Let ζ be any point in U. Let

$$\begin{aligned} \Psi(z) &= (z - \zeta)(1 - \zeta^* z)/z \\ &= -\zeta/z + (1 + |\zeta|^2) - \zeta^* z. \end{aligned} \tag{7}$$

Here, and throughout this paper, an asterisk on a complex quantity indicates the complex conjugate of that quantity. If $f \in S^*$ then

$$\begin{aligned} g(z) &= f(z)\Psi(z) \\ &= b_0 + b_1 z + b_2 z^2 + \cdots \end{aligned} \tag{8}$$

is univalent and weakly starlike as shown in [5]. The class of univalent weakly starlike functions clearly consists of functions mapping U onto starshaped domains and hence forms a subclass of K. However, "clearly" is not the same as a proof. So let us prove

LEMMA. 1. *Let $f \in S^*$ and let Ψ be defined by* (7). *Then* $g(z) = f(z)\Psi(z)$ *is close-to-convex with respect to f.*

PROOF. Let $r<1$. Define $f_r(z) = f(rz)/r$ and $g_r(z) = f_r(z)\Psi(z)$. Then $f_r(z)$ and $g_r(z)$ are regular for $|z| \leqq 1$. The same is true for the function

$$zg_r'(z)/f_r(z) = (zf_r'(z)/f_r(z))\Psi(z) + z\Psi'(z). \tag{9}$$

Further, as $r \to 0$, $f_r(z) \to f(z)$, $g_r(z) \to g(z)$, and $zg_r'(z)/f_r(z) \to zg'(z)/f(z)$, all uniformly in compact subsets of U.

Since $\Psi(z)$ is real and $z\Psi'(z) = \zeta/z - \zeta^* z$ is purely imaginary on $|z| = 1$, and $zf_r'(z)/f_r(z) = rzf'(rz)/f(rz)$ has positive real part on $|z| = 1$, it follows from (9) that $\operatorname{Re}\{zg_r'(z)/f_r(z)\} \geqq 0$ for all $z \in U$. Therefore $\operatorname{Re}\{zg'(z)/f(z)\} \geqq 0$ in U also. A simple calculation shows that $zg'(z)/f(z) = 1 - a_2\zeta + |\zeta|^2$ at $z = 0$ and since $|a_2| \leqq 2$, it follows that $\operatorname{Re}\{zg'(z)/f(z)\} > 0$ at $z = 0$. Hence $\operatorname{Re}\{zg'(z)/f(z)\} > 0$ for all $z \in U$ and the lemma is proved.

3. Once the above lemma has been proved, the remainder of the results follow quite easily. We prove first

THEOREM. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be in the class S^*. Then*

$$n - a_n = \gamma_n(2 - a_n) \tag{10}$$

where

$$\gamma_n = \sum_{\nu=1}^{n-1} k_{n,\nu}\mu_\nu. \tag{11}$$

In this expression the μ_ν *are complex constants depending on* f *but satisfying*

$$\mu_1 = 1, \quad |\mu_\nu| \leqq 1, \qquad \nu = 2, 3, \cdots. \tag{12}$$

The $k_{n,\nu}$ *are constants independent of* f *defined by*

$$k_{n,\nu} = \nu(n - \nu) \qquad n = 2, 3, \cdots; \nu = 1, 2, \cdots, n - 1. \tag{13}$$

Finally, for every n

$$|\gamma_n| \leqq n(n - 1)(n + 1)/6. \tag{14}$$

PROOF. From the lemma, the function $g(z)$ defined by (8) is close-to-convex. It therefore satisfies (6). From (8),

$$b_1 = -\zeta a_2 + (1 + |\zeta|^2),$$
$$b_n = -\zeta a_{n+1} + (1 + |\zeta|^2)a_n - \zeta^* a_{n-1}, \qquad n = 2, 3, \cdots.$$

In these formulas we may let $a_1 = 1$ and $a_0 = 0$. Putting these values into (6) and dividing both sides by ζ gives

$$|a_{n+1} - (1/\zeta + \zeta^*)a_n + (\zeta^*/\zeta)a_{n-1}| \leqq n|a_2 - (1/\zeta + \zeta^*)|.$$

This is true for every $n \geqq 1$ and each $\zeta \in U$. Letting $\zeta \to 1$, we have in the limit that

$$|a_{n+1} - 2a_n + a_{n-1}| \leqq n|a_2 - 2|, \qquad n = 1, 2, 3, \cdots.$$

This same result is equivalent to stating that, given $f \in S^*$, there exists a sequence of constants μ_ν satisfying (12) such that

$$a_{n+1} - 2a_n + a_{n-1} = n\mu_n(a_2 - 2). \tag{15}$$

Formula (10) is obvious in the case $n = 2$. It follows, in general, by induction. Assuming (10) true for n and $n - 1$, it is easily verified from (15) that

$$\begin{aligned}(n + 1) - a_{n+1} &= 2(n - a_n) - [(n - 1) - a_{n-1}] + n\mu_n(2 - a_2) \\ &= \left[\sum_{\nu=1}^{n-1} (2k_{n,\nu} - k_{n-1,\nu})\mu_\nu - n\mu_n\right](2 - a_2) \\ &= \left[\sum_{\nu=1}^{n} k_{n+1,\nu}\mu_\nu\right](2 - a_2).\end{aligned}$$

This proves (10).

Inequality (14) follows from (11) and the fact that

$$\sum_{\nu=1}^{n-1} k_{n,\nu} = \sum_{\nu=1}^{n-1} \nu(n-\nu) = n(n-1)(n+1)/6$$

as easily verified by induction.

4. Inequality (14) is sharp in the sense that no better numerical estimate is possible. This follows from the following

LEMMA 2. *Let $\kappa = e^{i\alpha}$, and set*

$$h(z) = z/(1-\kappa z)(1-\kappa^* z) = z + a_2 z^2 + a_3 z^3 + \cdots. \tag{16}$$

Then $h(z) \in S^*$ and is such that for each $n \geqq 2$

$$\lim_{\alpha\to 0} \frac{n-a_n}{2-a_2} = n(n-1)(n+1)/6. \tag{17}$$

PROOF. From (16)

$$\begin{aligned} a_2 &= \kappa + \kappa^* = 2\cos\alpha \\ a_n &= \kappa^{n-1} + \kappa^*\kappa^{n-2} + \cdots + \kappa^{*^{n-2}}\kappa + \kappa^{*^{n-1}} \\ &= (\kappa^n - \kappa^{*^n})/(\kappa - \kappa^*) \\ &= \sin n\alpha/\sin\alpha. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{n-a_n}{2-a_2} &= \frac{n\sin\alpha - \sin n\alpha}{2\sin\alpha(1-\cos\alpha)} \\ &= \frac{n\alpha - (n/3!)\alpha^3 - n\alpha + (n^3/3!)\alpha^3 + o(\alpha^3)}{2[\alpha + o(\alpha)][\alpha^2/2! + o(\alpha^2)]} \\ &= n(n-1)(n+1)/6 + o(1) \qquad \text{as} \quad \alpha \to 0, \end{aligned}$$

which is the desired result (17).

Since the starlike functions form a subclass of the normalized univalent functions, the results of this paper give $n(n-1)(n+1)/6$ as a lower bound for the constants c_n and d_n in (1) and (2). The writer is convinced that the constants d_n are actually much larger than this (assuming that they exist) and feels that it is probable that the c_n are larger also, but has no firm evidence on which to base this assertion.

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