RIEMANNIAN MANIFOLDS OF CONSTANT k-NULLITY[1]

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1. **Introduction.** The purpose of this note is to derive curvature conditions that will guarantee the existence of a product structure for a Riemannian manifold of constant k-nullity. The proof is modeled after similar theorems for Riemannian and Kähler manifolds of constant nullity [5], [6]. Nullity was defined by Chern and Kuiper [1]. Ôtsuki defined the concept of nullity relative to a constant k, so that nullity became the special case $k=0$ [4]. A definition in terms of vectors was given by Gray, who also shortened the name to k-nullity [2].

2. **Definitions and the main theorem.** Let M_m denote the tangent space to the Riemannian manifold M at the point m, and let R_{xy} denote the curvature transformation associated with x, $y \in M_m$.

DEFINITION. Let $B_{xy}z = R_{xy}z - k\{\langle x, z\rangle y - \langle y, z\rangle x\}$, where x, y, $z \in M_m$ and k is a constant.

Then B is a tensor of the same type as R, and B possesses the symmetries of R, [2].

DEFINITION. Let $N_k(m) = \{z \in M_m : B_{xy}z = 0$ for all x, $y \in M_m\}$.

$N_k(m)$ is called the k-nullity space at m. The dimension $\mu(m)$ of $N_k(m)$ is the k-nullity at m. The conullity space $C_k(m)$ is the orthogonal complement to the nullity space at m. Elements of $C_k(m)$ are called conullity vectors. A conullity plane is a plane spanned by conullity vectors.

THEOREM. *Let M^n be a complete, connected, and simply connected C^∞ Riemannian manifold of constant k-nullity μ, where $0 < \mu \leqq n-3$. If $n-\mu$ is odd and the sectional curvatures of all conullity planes are unequal to k, then M^n is a direct metric product, $M^n = K^\mu \times C^{n-\mu}$, where K^μ and $C^{n-\mu}$ are complete, and K^μ has constant curvature k.*

3. **Proof of the theorem.** If μ is constant and positive, the distribution of k-nullity spaces is integrable, and the integral manifolds are complete submanifolds of M^n of constant curvature k, [2]. Any one of these integral manifolds provides one factor for a product structure of M^n.

DEFINITION. For each $u \in N_k(m)$ and $x \in C_k(m)$, let $T_u(x) = \mathrm{P}(\nabla_x U)$,

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where P is the projection of M_m^n into $C_k(m)$ and U is any nullity extension of u.

T_u is a well-defined linear operator on $C_k(m)$, called a conullity operator [6]. The nonvanishing of the conullity operators represents the obstruction to the existence of a product structure for M^n, for if each conullity operator is zero, we can apply DeRham's decomposition theorem to obtain the theorem [5].

LEMMA (THE CONULLITY IDENTITY). *If T is a conullity operator at m, then*

$$\mathfrak{S}_{x,y,z}B_{xy}(T(z)) = 0 \qquad \text{for all } x, y, z \in C_k(m).$$

PROOF. Let T be the conullity operator associated with $u \in N_k(m)$. The second Bianchi identity for B states that $\mathfrak{S}_{x,y,z} \nabla_x(B)_{yz}(u) = 0$. Using the definition of B in terms of R, and the relation $\nabla_X(B)_{YZ}(u) = \nabla_X(B_{YZ}u) - B_{\nabla_X Y,Z}u - B_{Y,\nabla_X Z}u - B_{Y,Z}(\nabla_X u)$, where X, Y, Z, and U are extensions of x, y, z and u, with U a k-nullity field, we find that

$$0 = \mathfrak{S}_{x,y,z}B_{yz}(\nabla_x U) = \mathfrak{S}_{x,y,z}B_{yz}(T(x)).$$

REMARK. Although this identity is valid for all values of μ, it is nontrivial only when there are at least three independent conullity vectors. This is the reason for the $n-\mu \leqq 3$ hypothesis in the theorem.

LEMMA. *If λ is a real eigenvalue of a conullity operator, then λ is zero.*

PROOF. Let T be the conullity operator at m associated with $u \in N_k(m)$. We may assume that u is a unit vector because T is linear in u. As in Theorem (3.1) of [5], we calculate the curvature of M^n along a unit speed geodesic σ starting at m in the u direction. The frame field used in the calculation remains valid for this case [3]. If $P(t)$ is the matrix of $T_{\sigma'(t)}$ relative to the adapted frame field used in this calculation, we obtain a differentiable matrix-valued function P that satisfies the differential equation $P' = -P^2 - kI$. Since M^n is complete, the domain of P is the entire real line.

Let x be an eigenvector of $P(0)$ with the real eigenvalue λ. The relation $P' = -P^2 - kI$ implies that x is an eigenvector of any derivative of P at time zero. Using the power series representation of P given by Picard iteration, we can deduce that x is an eigenvector of $P(t)$ for all t. Thus, we may assume that $P_{j1}(t) = 0$ for $j \neq 1$. If we set $p(t) = P_{11}(t)$, we find that p satisfies the equation $p' = -p^2 - k$.

We can assume that $k \neq 0$, as this case is solved in Theorem (3.1) of [5].

Thus, if $k < 0$, $p(t) = \omega(p_0 + \omega \tanh \omega t)/(\omega + p_0 \tanh \omega t)$, where $\omega = \sqrt{-k}$, and $p_0 = p(0)$.

If $k>0$, $p(t)=\omega(p_0-\omega \tan \omega t)/(\omega+p_0 \tan \omega t)$, where $\omega=\sqrt{+k}$.

In either case, if $p_0\neq 0$, the denominator of p would vanish for some value of t, and p would not be differentiable. Thus, $\lambda=p_0=0$.

To show that each conullity operator T vanishes, it suffices to show that the eignevalues of T are real and that T can have no multiple eigenspaces with eigenvalue zero. The proofs of these facts are algebraic in nature, and are similar to Theorems (4.2) and (4.6) of [5], which used the conullity identity for R and T, the symmetries of R, and the fact that the sectional curvatures of conullity planes were nonzero. In this case, we have the conullity identity for B and T, the fact that B shares the symmetries of R, and the fact that $\langle B_{xy}x, y\rangle \neq 0$ for all $x, y, \in C_k(m)$.

REMARK. It should also be clear that a theorem analogous to Theorem (2*) of [5] holds. That is, if we replace the hypotheses that $n-\mu$ is odd, and that the sectional curvatures of conullity planes are unequal to k, by the condition that the tensor B is positive or negative definite when restricted to pairs of conullity vectors, then the conclusion of the theorem holds. This is again an algebraic consequence of the theorems in [5].

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