AN INVERSION THEOREM FOR HANKEL TRANSFORMS[1]

ALAN L. SCHWARTZ[2]

It is a well-known fact of classical Fourier analysis that if f is a function integrable on the real line and of bounded variation in a neighborhood of x, then

$$(1)\qquad \lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} e^{iux}du\int_{-\infty}^{\infty} f(y)e^{-iuy}dy = (1/2)\{f(x+0)+f(x-0)\}.$$

Analogous results hold for other integral transforms. It is our intention to study the behaviour of a similar inversion formula for the Hankel transforms defined below.

Let ν be a fixed real number exceeding $(-1/2)$ and let L consist of all functions measurable on $0<x<\infty$ such that

$$\|f\| = \int_0^{\infty} |f(x)|\, dm(x) < \infty$$

where

$$dm(x) = [2^{\nu}\Gamma(\nu+1)]^{-1}x^{2\nu+1}dx.$$

Let

$$\mathcal{J}(x) = 2^{\nu}\Gamma(\nu+1)x^{-\nu}J_{\nu}(x),$$

where J_ν is the Bessel function of the first kind of order ν. We are interested in whether the following formula analogous to (1) holds

$$(2)\qquad \lim_{\lambda\to\infty}\int_0^{\lambda}\mathcal{J}(xu)dm(u)\int_0^{\infty} f(y)\mathcal{J}(uy)dm(y) = (1/2)\{f(x+0)+f(x-0)\}.$$

Equation (2) does not hold under circumstances as general as those for which (1) holds. What is needed is a restriction on the behaviour of f near 0; we shall prove the following theorem after a few remarks.

THEOREM. *Suppose f is in L and*

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$$\int_0^1 |f(y)|\, y^{\nu+(1/2)} dy < \infty, \tag{3}$$

then if $x>0$ *and if* f *is of bounded variation in a neighborhood of* x, (2) *holds.*

Finally we will show by means of an example that the exponent $\nu+(1/2)$ in (3) cannot be increased.

We will need the following well-known properties of Bessel functions

$$J_\mu(x) = (2/\pi x)^{(1/2)} \cos(x - (1/2)\mu\pi - (1/4)\pi) + O(x^{-(3/2)}) \quad \text{as } x \to \infty, \tag{4}$$

in particular

$$J_\mu(x) = O(1/\sqrt{x}) \qquad \text{as } x \to \infty, \tag{5}$$

$$J_\mu(x) = O(x^\mu) \qquad \text{as } x \to 0, \tag{6}$$

$$\int_0^\lambda J_\mu(uy)J_\mu(ux)udu = \lambda(x^2 - y^2)^{-1}\{xJ_{\nu+1}(\lambda x)J_\nu(\lambda y) - yJ_{\nu+1}(\lambda y)J_\nu(\lambda x)\} \tag{7}$$

(see [**5** p. 134]).

Proof of the Theorem. The integral in (2) can be written as a sum of two integrals with y in the ranges $(0, \delta)$ and (δ, ∞) where $0<\delta<x$. The proof of Hankel's theorem in [**4**, pp. 240–242] is easily adapted to show

$$\lim_{\lambda\to\infty} \int_0^\lambda \mathcal{J}(xu)dm(u) \int_\delta^\infty f(y)\mathcal{J}(uy)dm(y) = (1/2)\{f(x+0) + f(x-0)\};$$

condition (3) does not enter that argument. Thus it suffices to show

$$\lim_{\lambda\to\infty} \int_0^\lambda \mathcal{J}(xu)dm(u) \int_0^\delta f(y)\mathcal{J}(uy)dm(y) = 0.$$

By Fubini's theorem and (7) this last integral is equal to

$$\lambda x^{1-\nu} J_{\nu+1}(\lambda x) \int_0^\delta f(y)(x^2 - y^2)^{-1} y^{\nu+1} J_\nu(\lambda y)dy - \lambda x^{-\nu} J_\nu(\lambda x) \int_0^\delta f(y)(x^2 - y^2)^{-1} y^{\nu+2} J_\nu(\lambda y)dy.$$

We will show that the first integral is $o(1/\sqrt{\lambda})$ as $\lambda\to\infty$. The same methods serve to analyze the second integral and the theorem will be proved.

From (6) we have

$$\begin{aligned}\left|\int_0^{(1/\lambda)} f(y)(x^2-y^2)^{-1}J_\nu(\lambda y)y^{\nu+1}dy\right| \\ &= O(\lambda^\nu)\int_0^{(1/\lambda)} |f(y)|\, y^{2\nu+1}dy \\ &= O(1/\sqrt{\lambda})\int_0^{(1/\lambda)} |f(y)|\, y^{\nu+(1/2)}dy = o(1/\sqrt{\lambda})\end{aligned}$$

by (3). For $y \geqq (1/\lambda)$ we use the asymptotic expansion (4). The cosine term contributes

$$\left(\frac{2}{\pi\lambda}\right)^{(1/2)}\int_{(1/\lambda)}^{\delta} f(y)(x^2-y^2)^{-1}\cos(\lambda y-(1/2)\nu\pi-(1/4)\pi)y^{\nu+(1/2)}dy = o(1/\sqrt{\lambda})$$

by the classical Riemann-Lebesgue lemma. Finally the O term is estimated in two parts:

$$\begin{aligned}\left|\int_{(1/\lambda)}^{(1/\sqrt{\lambda})} f(y)(x^2-y^2)^{-1}O((\lambda y)^{-(3/2)})y^{\nu+1}dy\right| \\ &= O(1/\sqrt{\lambda})\int_{(1/\lambda)}^{(1/\sqrt{\lambda})} |f(y)|\,(\lambda y)^{-1}y^{\nu+(1/2)}dy \\ &= O(1/\sqrt{\lambda})\int_{(1/\lambda)}^{(1/\sqrt{\lambda})} |f(y)|\, y^{\nu+(1/2)}dy = o(1/\sqrt{\lambda})\end{aligned}$$

by (3); and

$$\begin{aligned}\left|\int_{(1/\sqrt{\lambda})}^{\delta} f(y)(x^2-y^2)^{-1}O((\lambda y)^{-(3/2)})y^{\nu+1}dy\right| \\ &= O(1/\lambda)\int_{(1/\sqrt{\lambda})}^{\delta} |f(y)|\,(y\sqrt{\lambda})^{-1}y^{\nu+(1/2)}dy = O(1/\lambda) = o(1/\sqrt{\lambda})\end{aligned}$$

by (3). Q.E.D.

We now give an example that shows that the exponent $\nu+(1/2)$ in (3) cannot be increased. Let

$$\begin{aligned} f(y) &= y^{-(\nu+(3/2))} && 0 < y \leqq 1 \\ &= 0 && y > 1 \end{aligned}$$

and let $x>1$. Let

$$I_\lambda = \int_0^\lambda \mathcal{J}(xu)dm(u) \int_0^\infty f(y)\mathcal{J}(uy)dm(y).$$

Thus (2) holds if and only if

$$I_\lambda \to 0 \qquad \text{as } \lambda \to \infty.$$

We will find a sequence λ_i such that

$$\lambda_i \to \infty \qquad \text{as } i \to \infty,$$

and such that for some positive constant C

$$|I_{\lambda_i}| > C \qquad (i = 1, 2, 3, \cdots).$$

Let $x_1, x_2, x_3, \cdots$ be the positive real zeros of $J_\nu(x)$ in ascending order and let $\lambda_i = x_i/x$. Then by (7) and Fubini's theorem

$$\begin{aligned} I_{\lambda_i} &= \lambda_i x^{1-\nu} J_{\nu+1}(x_i) \int_0^1 \frac{J_\nu(\lambda_i y)}{(x^2 - y^2)y^{1/2}}\, dy, \\ &= x^{1-\nu} \lambda_i^{1/2} J_{\nu+1}(x_i) \int_0^{\lambda_i} \frac{J_\nu(w)}{[x^2 - (w/\lambda_i)^2]w^{1/2}}\, dw. \end{aligned}$$

It can be easily shown that

$$\lim_{i\to\infty} \int_0^{\lambda_i} \frac{J_\nu(w)dw}{[x^2 - (w/\lambda_i)^2]w^{1/2}} = x^{-2} \int_0^\infty J_\nu(w)w^{-1/2}dw.$$

This last expression has the value

$$\Gamma((2\nu + 1)/4)/2^{1/2}\Gamma((2\nu + 3)/4)$$

(see [2, p. 22, formula (7)]).

From (4) it follows that

$$\begin{aligned} J_{\nu+1}(x) &= (2/\pi x)^{1/2} \cos(x - \beta - (\pi/2)) + O(x^{-3/2}) \\ &= (2/\pi x)^{1/2} \sin(x - \beta) + O(x^{-3/2}) \end{aligned}$$

where $\beta = (2\nu + 1)\pi/4$ and

$$J_\nu(x) = (2/\pi x)^{1/2} \cos(x - \beta) + O(x^{-3/2}).$$

Since $J_\nu(x_i) = 0$, we see that

$$|J_{\nu+1}(x_i)| \geqq (\pi x_i)^{-1/2}$$

for i sufficiently large. Thus for some constant C we have

$$x^{1-\nu}\lambda_i^{1/2} \left| J_{\nu+1}(x_i) \right| \geqq C$$

and so

$$\left| I_{\lambda_i} \right| \geqq x^{\nu-1}C.$$

Thus, it follows that $\nu+(1/2)$ is indeed the largest exponent in (3) for which the theorem holds.

REMARK. Suppose $\nu=(n-2)/2$ where n is an integer greater than 1, and suppose f is defined on R_n, Euclidean n-space. f is radial if there is a function g defined on $(0, \infty)$ such that

$$f(\mathbf{x}) = g(|\mathbf{x}|)$$

for almost all $\mathbf{x}$ in R_n. Then f is integrable if and only if g is in L; furthermore [1, p. 69] $\int_0^\infty g(y)\mathcal{J}(uy)\, dm(u)$ is essentially the Fourier transform of f at $\mathbf{y}$ for any point $\mathbf{y}$ such that $|\mathbf{y}| = u$. Then our theorem says that the multiple Fourier transform of f can be inverted by spherical sums if $f(\mathbf{x})|\mathbf{x}|^{(n-1)/2}$ is integrable and our example yields one of a function supported on $|\mathbf{x}| \leqq 1$ for which localization fails to hold for the spherical sums.

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REFERENCES

1. S. Bochner and K. Chandrasekharan, *Fourier transforms*, Princeton Univ. Press, Princeton, N. J., 1949.

2. Erdélyi et al, *Tables of integral transforms*, Vol. II, McGraw-Hill, New York, 1953.

3. I. I. Hirschman, Jr., *Variation diminishing Hankel transforms*, J. Analyse Math. 8 (1960/1961), 307–336.

4. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford Univ. Press, London, 1959.

5. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, London, 1966.

UNIVERSITY OF MISSOURI