

AN INVERSION THEOREM FOR HANKEL TRANSFORMS¹

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It is a well-known fact of classical Fourier analysis that if f is a function integrable on the real line and of bounded variation in a neighborhood of x , then

$$(1) \quad \lim_{\lambda \rightarrow \infty} \int_{-\lambda}^{\lambda} e^{iux} du \int_{-\infty}^{\infty} f(y) e^{-iuy} dy = (1/2) \{f(x+0) + f(x-0)\}.$$

Analogous results hold for other integral transforms. It is our intention to study the behaviour of a similar inversion formula for the Hankel transforms defined below.

Let ν be a fixed real number exceeding $(-1/2)$ and let L consist of all functions measurable on $0 < x < \infty$ such that

$$\|f\| = \int_0^{\infty} |f(x)| dm(x) < \infty$$

where

$$dm(x) = [2^{\nu} \Gamma(\nu + 1)]^{-1} x^{2\nu+1} dx.$$

Let

$$g(x) = 2^{\nu} \Gamma(\nu + 1) x^{-\nu} J_{\nu}(x),$$

where J_{ν} is the Bessel function of the first kind of order ν . We are interested in whether the following formula analogous to (1) holds

$$(2) \quad \lim_{\lambda \rightarrow \infty} \int_0^{\lambda} g(xu) dm(u) \int_0^{\infty} f(y) g(uy) dm(y) = (1/2) \{f(x+0) + f(x-0)\}.$$

Equation (2) does not hold under circumstances as general as those for which (1) holds. What is needed is a restriction on the behaviour of f near 0; we shall prove the following theorem after a few remarks.

THEOREM. *Suppose f is in L and*

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$$(3) \quad \int_0^1 |f(y)| y^{\nu+(1/2)} dy < \infty,$$

then if $x > 0$ and if f is of bounded variation in a neighborhood of x , (2) holds.

Finally we will show by means of an example that the exponent $\nu + (1/2)$ in (3) cannot be increased.

We will need the following well-known properties of Bessel functions

$$(4) \quad J_\mu(x) = (2/\pi x)^{(1/2)} \cos(x - (1/2)\mu\pi - (1/4)\pi) + O(x^{-(3/2)})$$

as $x \rightarrow \infty$,

in particular

$$(5) \quad J_\mu(x) = O(1/\sqrt{x}) \quad \text{as } x \rightarrow \infty,$$

$$(6) \quad J_\mu(x) = O(x^\mu) \quad \text{as } x \rightarrow 0,$$

$$(7) \quad \int_0^\lambda J_\mu(uy) J_\mu(ux) u du = \lambda(x^2 - y^2)^{-1} \{ x J_{\nu+1}(\lambda x) J_\nu(\lambda y) - y J_{\nu+1}(\lambda y) J_\nu(\lambda x) \}$$

(see [5 p. 134]).

PROOF OF THE THEOREM. The integral in (2) can be written as a sum of two integrals with y in the ranges $(0, \delta)$ and (δ, ∞) where $0 < \delta < x$. The proof of Hankel's theorem in [4, pp. 240-242] is easily adapted to show

$$\lim_{\lambda \rightarrow \infty} \int_0^\lambda \mathcal{G}(xu) dm(u) \int_\delta^\infty f(y) \mathcal{G}(uy) dm(y) = (1/2) \{ f(x+0) + f(x-0) \};$$

condition (3) does not enter that argument. Thus it suffices to show

$$\lim_{\lambda \rightarrow \infty} \int_0^\lambda \mathcal{G}(xu) dm(u) \int_0^\delta f(y) \mathcal{G}(uy) dm(y) = 0.$$

By Fubini's theorem and (7) this last integral is equal to

$$\lambda x^{1-\nu} J_{\nu+1}(\lambda x) \int_0^\delta f(y) (x^2 - y^2)^{-1} y^{\nu+1} J_\nu(\lambda y) dy - \lambda x^{-\nu} J_\nu(\lambda x) \int_0^\delta f(y) (x^2 - y^2)^{-1} y^{\nu+2} J_\nu(\lambda y) dy.$$

We will show that the first integral is $o(1/\sqrt{\lambda})$ as $\lambda \rightarrow \infty$. The same

methods serve to analyze the second integral and the theorem will be proved.

From (6) we have

$$\begin{aligned} \left| \int_0^{(1/\lambda)} f(y)(x^2 - y^2)^{-1} J_\nu(\lambda y) y^{\nu+1} dy \right| \\ = O(\lambda^\nu) \int_0^{(1/\lambda)} |f(y)| y^{2\nu+1} dy \\ = O(1/\sqrt{\lambda}) \int_0^{(1/\lambda)} |f(y)| y^{\nu+(1/2)} dy = o(1/\sqrt{\lambda}) \end{aligned}$$

by (3). For $y \geq (1/\lambda)$ we use the asymptotic expansion (4). The cosine term contributes

$$\begin{aligned} \left(\frac{2}{\pi\lambda} \right)^{(1/2)} \int_{(1/\lambda)}^\delta f(y)(x^2 - y^2)^{-1} \cos(\lambda y - (1/2)\nu\pi - (1/4)\pi) y^{\nu+(1/2)} dy \\ = o(1/\sqrt{\lambda}) \end{aligned}$$

by the classical Riemann-Lebesgue lemma. Finally the O term is estimated in two parts:

$$\begin{aligned} \left| \int_{(1/\lambda)}^{(1/\sqrt{\lambda})} f(y)(x^2 - y^2)^{-1} O((\lambda y)^{-(3/2)}) y^{\nu+1} dy \right| \\ = O(1/\sqrt{\lambda}) \int_{(1/\lambda)}^{(1/\sqrt{\lambda})} |f(y)| (\lambda y)^{-1} y^{\nu+(1/2)} dy \\ = O(1/\sqrt{\lambda}) \int_{(1/\lambda)}^{(1/\sqrt{\lambda})} |f(y)| y^{\nu+(1/2)} dy = o(1/\sqrt{\lambda}) \end{aligned}$$

by (3); and

$$\begin{aligned} \left| \int_{(1/\sqrt{\lambda})}^\delta f(y)(x^2 - y^2)^{-1} O((\lambda y)^{-(3/2)}) y^{\nu+1} dy \right| \\ = O(1/\lambda) \int_{(1/\sqrt{\lambda})}^\delta |f(y)| (y\sqrt{\lambda})^{-1} y^{\nu+(1/2)} dy = O(1/\lambda) = o(1/\sqrt{\lambda}) \end{aligned}$$

by (3). Q.E.D.

We now give an example that shows that the exponent $\nu + (1/2)$ in (3) cannot be increased. Let

$$\begin{aligned} f(y) &= y^{-(\nu+(3/2))} & 0 < y \leq 1 \\ &= 0 & y > 1 \end{aligned}$$

and let $x > 1$. Let

$$I_\lambda = \int_0^\lambda g(xu) dm(u) \int_0^\infty f(y) g(uy) dm(y).$$

Thus (2) holds if and only if

$$I_\lambda \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty.$$

We will find a sequence λ_i such that

$$\lambda_i \rightarrow \infty \quad \text{as } i \rightarrow \infty,$$

and such that for some positive constant C

$$|I_{\lambda_i}| > C \quad (i = 1, 2, 3, \dots).$$

Let x_1, x_2, x_3, \dots be the positive real zeros of $J_\nu(x)$ in ascending order and let $\lambda_i = x_i/x$. Then by (7) and Fubini's theorem

$$\begin{aligned} I_{\lambda_i} &= \lambda_i x^{1-\nu} J_{\nu+1}(x_i) \int_0^1 \frac{J_\nu(\lambda_i y)}{(x^2 - y^2)y^{1/2}} dy, \\ &= x^{1-\nu} \lambda_i^{1/2} J_{\nu+1}(x_i) \int_0^{\lambda_i} \frac{J_\nu(w)}{[x^2 - (w/\lambda_i)^2]w^{1/2}} dw. \end{aligned}$$

It can be easily shown that

$$\lim_{i \rightarrow \infty} \int_0^{\lambda_i} \frac{J_\nu(w) dw}{[x^2 - (w/\lambda_i)^2]w^{1/2}} = x^{-2} \int_0^\infty J_\nu(w) w^{-1/2} dw.$$

This last expression has the value

$$\Gamma((2\nu + 1)/4)/2^{1/2}\Gamma((2\nu + 3)/4)$$

(see [2, p. 22, formula (7)]).

From (4) it follows that

$$\begin{aligned} J_{\nu+1}(x) &= (2/\pi x)^{1/2} \cos(x - \beta - (\pi/2)) + O(x^{-3/2}) \\ &= (2/\pi x)^{1/2} \sin(x - \beta) + O(x^{-3/2}) \end{aligned}$$

where $\beta = (2\nu + 1)\pi/4$ and

$$J_\nu(x) = (2/\pi x)^{1/2} \cos(x - \beta) + O(x^{-3/2}).$$

Since $J_\nu(x_i) = 0$, we see that

$$|J_{\nu+1}(x_i)| \geq (\pi x_i)^{-1/2}$$

for i sufficiently large. Thus for some constant C we have

$$x^{1-\nu} \lambda_i^{1/2} |J_{\nu+1}(x_i)| \geq C$$

and so

$$|I_{\lambda_i}| \geq x^{\nu-1} C.$$

Thus, it follows that $\nu + (1/2)$ is indeed the largest exponent in (3) for which the theorem holds.

REMARK. Suppose $\nu = (n-2)/2$ where n is an integer greater than 1, and suppose f is defined on R_n , Euclidean n -space. f is radial if there is a function g defined on $(0, \infty)$ such that

$$f(\mathbf{x}) = g(|\mathbf{x}|)$$

for almost all \mathbf{x} in R_n . Then f is integrable if and only if g is in L ; furthermore [1, p. 69] $\int_0^\infty g(y)g(uy) dm(u)$ is essentially the Fourier transform of f at \mathbf{y} for any point \mathbf{y} such that $|\mathbf{y}| = u$. Then our theorem says that the multiple Fourier transform of f can be inverted by spherical sums if $f(\mathbf{x})|\mathbf{x}|^{(n-1)/2}$ is integrable and our example yields one of a function supported on $|\mathbf{x}| \leq 1$ for which localization fails to hold for the spherical sums.

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