MONOTONE MAPPING PROPERTIES OF HEREDITARILY INFINITE DIMENSIONAL SPACES[1]

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1. **Introduction.** Since the discovery of hereditarily infinite-dimensional (HID) spaces by D. W. Henderson [**3**], questions have naturally arisen about the topological properties of such spaces. A hereditarily infinite-dimensional space is an infinite-dimensional compact metric space each of whose nondegenerate subcontinua is infinite dimensional.

In a previous paper [**7**], we studied the structure of HID spaces. In this paper, we consider the behavior of HID spaces under monotone mappings. The principal result of this paper is that, given an arbitrary compact metric space Y, there is an HID space X and a monotone map $f:X\to Y$. We also show that an arbitrary HID space can be mapped monotonically onto a space of any preassigned dimension, and that, given an HID space X and a positive integer n, there is an n-dimensional space Y and a monotone map $f: Y\to X$.

R. H. Bing showed in [**2**] that each nondegenerate monotone image of a pseudo-arc is a pseudo-arc. The results of this paper show that no similar monotone invariance property holds for spaces of dimension greater than 1.

In this paper, all spaces will be compact metric spaces (compacta). We will be dealing with the Hilbert cube, which we regard as being the product of a countably infinite collection of straight line intervals

$$I^\omega = I_1 \times I_2 \times I_3 \times \cdots, \quad \text{where } I_j = [-1/2^j, 1/2^j].$$

By the dimension of a space we will mean the Menger-Urysohn, or small inductive, definition of dimension, or any equivalent definition (see [**5** appendix]).

2. **Finite-dimensional monotone images of HID spaces.** Given an arbitrary HID space X, what can we say about a monotone image of X? A monotone image of X can have any preassigned finite dimension, as the following proposition shows. We include this proposition for completeness.

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PROPOSITION 1. *Let X be a compact metric space with* $\dim X \geqq n$. *Then there is an n-dimensional compact metric space X_n and a monotone map $f: X \to X_n$.*

PROOF. Since $\dim X \geqq n$, there is an essential map $g: X \to I^n$ [6, Theorem III. 5]. Let $f_L f$ be the Whyburn factorization of g [4, Theorem 3.40] and let $X_n = f(X)$. Since f_L is a uniformly zero-dimensional mapping, f_L does not lower dimension [6, Proposition III. 7(A)]; hence $\dim X_n \leqq n$. On the other hand, f_L is an essential map, for otherwise g would not be essential; hence [6, Theorem III. 5], $\dim X_n \geqq n$. The proposition is proved.

It should, perhaps, be remarked that the character of the space X_n is not at all clear. For example, there may well be points at which X_n has dimension less than n.

3. **An HID continuum which maps monotonically onto I^1.** Let C be the canonical Cantor set in I^1, and let f be the Cantor function on I^1 [4, p. 131]. Let $J_1, J_2, \cdots$ be the closures of the components of I^1-C in any convenient exhaustive order, and let $p_i = f(J_i)$. We will construct an HID continuum which maps monotonically onto I^1 by substituting an HID continuum X_i for J_i, the monotone map being that map obtained by sending X_i to p_i.

We regard I^1 as being embedded in I^ω as the first factor.

Let X be an HID continuum in I^ω from $(-1/2, 0, 0, \cdots)$ to $(1/2, 0, 0, \cdots)$ (see [7]).

Define a homeomorphism h_i of I^ω into I^ω by

$$h_i(x_1, x_2, x_3, \cdots,) = ((b_i - a_i)x_1 + (a_i + b_i)/2, (b_i - a_i)x_2, (b_i - a_i)x_3, \cdots)$$

where $[a_i, b_i] = J_i$. Then h_i takes I^1 linearly onto J_i and shrinks all other coordinates of I^ω linearly and proportionately. Let $X_i = h_i(X)$. Then X_i is an HID continuum joining the end points of J_i,

$$X_i \cap C = J_i \cap C = \{a_i, b_i\},$$

and $X_i \cap X_j = \varnothing$ if $i \neq j$.

Let $K = C \cup \bigcup_{i=1}^\infty X_i$. K is the desired HID continuum, as we shall prove in Lemma 1. Figure 1 gives an indication of what the projection of K on $I_1 \times I_2$ might look like. We remark that K has dimension 1 at each inaccessible point of C.

LEMMA 1. *K is a compact HID continuum which can be mapped monotonically onto I^1.*

PROOF. Any infinite sequence of points in any one X_i has a limit

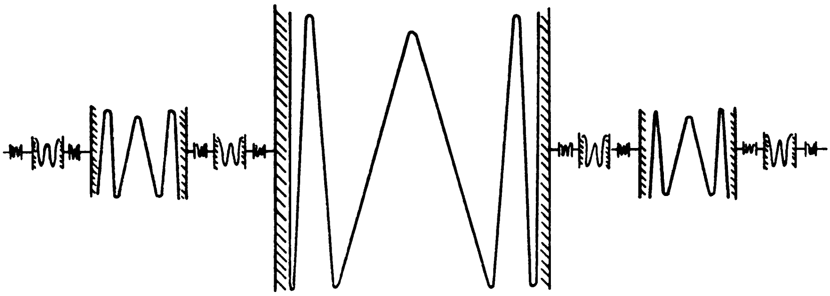

FIGURE 1

point in that X_i. Since the diameters of the X_i's tend to zero as $i\to\infty$, and each X_i contains a point of C, any infinite sequence of points in an infinite number of the X_i has a limit point in C. Thus K is compact.

Since each X_i is a continuum containing the end points of J_i, K is connected. Any nondegenerate subcontinuum of K must contain a nondegenerate subcontinuum of some X_i, and must therefore be infinite dimensional. This shows that K is HID. Finally, the monotone map of K onto I^1 is the map mentioned earlier in this section.

If X is chosen to be hereditarily indecomposable, then the X_i's are the smallest possible preimages of points under a monotone map of K onto I^1, since the only hereditarily indecomposable subcontinua of I^1 are points.

4. **An HID continuum which maps monotonically onto I^ω.** We will now construct an HID continuum K_∞ which maps monotonically onto I^ω. In this construction, the basic building block will be the HID continuum K of §3.

The plan is to obtain a countable collection of HID spaces, each of which lies in I^ω and is the cartesian product of a Cantor set with a homeomorphic copy of K. These spaces will be constructed so that their union is an HID continuum which maps monotonically onto a Hilbert cube.

We first single out a countable collection $\{H_i\}$ of Hilbert cubes each of which is properly embedded in I^ω. Let p_i denote the ith prime number, and let H_i denote the Hilbert cube given by

$$H_i = I_{p_i} \times I_{p_i^2} \times I_{p_i^3} \times \cdots \times I_{p_i^k} \times \cdots .$$

Let $H = H_1 \times H_2 \times \cdots$; H is the product of a countable collection of Hilbert cubes and is itself a Hilbert cube. We regard H as being a subset of I^ω where all factors of H which are not specified are assumed to be $\{0\}$. (This convention will be used in the following discussion without further comment.)

We will also need to single out the Hilbert cube $H_0 = I_2 \times I_3 \times I_5 \times \cdots \times I_{p_k} \times \cdots$ which is a subcube of H.

Now let K_i be a copy of K constructed in the Hilbert cube H_i, and let C_i be the canonical Cantor set in I_{p_i}. Let $M_i = C_1 \times C_2 \times C_3 \times \cdots \times C_{i-1} \times K_i \times C_{i+1} \times \cdots$; M_i is just the cartesian product of a Cantor set with K_i. Observe that M_i contains the Cantor set $C_0 = C_1 \times C_2 \times C_3 \times \cdots$ since $C_i \subset K_i$.

We define K_∞ to be $\bigcup_{i=1}^\infty M_i$.

THEOREM 1. *K_∞ is a compact, hereditarily infinite-dimensional continuum which can be mapped monotonically onto the Hilbert cube.*

PROOF. (1) K_∞ is compact: K_i is compact by Lemma 1, hence M_i is compact, being the product of compact spaces. Any finite union of the M_i is compact, since each M_i is. Let $\{x_j\}$ be a sequence of points with each x_j belonging to a different M_i. Since the diameter of H_i is less than 2^{2-p_i}, it follows that the diameter of M_i is less than 2^{2-p_i}. Thus if $x_j \in M_k$, it follows that there is a point $y_j \in C_0$ such that $d(x_j, y_j) < 2^{2-p_k}$. Since C_0 is compact, the sequence $\{y_j\}$ has a limit point $y_0 \in C_0$; and it follows that some subsequence of the sequence $\{x_j\}$ also converges to y_0. But by construction, $C_0 \subset K_\infty$; hence K_∞ is compact.

(2) K_∞ is connected: Let $x, y \in K_\infty$. We will construct a continuum in K_∞ containing both x and y. We may assume without loss of generality that both x and y belong to C_0.

Let $x = (x_1, x_2, \cdots)$, $x_i \in C_i \subset H_i$, $y = (y_1, y_2, \cdots)$, $y_i \in C_i \subset H_i$. In M_i, there is a continuum containing both $(y_1, y_2, \cdots, y_{i-1}, x_i, x_{i+1}, \cdots)$ and $(y_1, y_2, \cdots, y_{i-1}, y_i, x_{i+1}, \cdots)$; namely,

$$y_1 \times y_2 \times \cdots \times y_{i-1} \times K_i \times x_{i+1} \times \cdots.$$

Call this L_i. Let $L = \bigcup_{i=1}^\infty L_i$. Then L is a compact set which contains both x and y; L will be shown to be connected when we show that $L_i \cap L_{i+1} \neq \emptyset$ for all i. But by construction we have

$$(y_1, y_2, \cdots, y_{i-1}, y_i, x_{i+1}, \cdots) \in L_i$$

and

$$(y_1, y_2, \cdots, y_{(i+1)-1}, x_{i+1}, x_{(i+1)+1}, \cdots) \in L_{i+1}.$$

Connectedness of L implies connectedness of K_∞.

(3) K_∞ is hereditarily infinite dimensional: Observe first that C_0 is 0-dimensional.

Let Y be a nondegenerate subcontinuum of K_∞, and let $x \in Y - C_0$. Then $x \in M_j$ for some integer j, and x_j, the jth coordinate of x, does not belong to C_j. Then there is a neighborhood U of x such that for any $y \in U$, $Y_j \notin C_j$. Now if $p \in M_i$, $i \neq j$, it follows from the construction that $p_j \in C_j$. Thus x is not a limit point of $\bigcup_{i \neq j} M_i$.

This implies that x belongs to a nondegenerate subcontinuum Y' of M_i, and Y' must lie in a copy of K_i, since M_i is a Cantor set of copies of K_i. Then Lemma 1 implies that Y' is infinite dimensional; hence Y is also infinite dimensional.

(4) There is a monotone map of K_∞ onto H_0: Let ϕ_n be the Cantor map on I_n. Define $f: K_\infty \to H_0$ by

$$\begin{aligned} f(x) &= f(x_2, x_{2^2}, x_{2^3}, \cdots, x_3, x_{3^2}, x_{3^3}, \cdots, x_5, x_{5^2}, x_{5^3}, \cdots, \cdots) \\ &= (\phi(x_2), 0, 0, \cdots, \phi(x_3), 0, 0, \cdots, \phi(x_5), 0, 0, \cdots, \cdots). \end{aligned}$$

f is onto, since $C_0 \subset K_\infty$ and $f: C_0 \to H_0$ is onto. In fact, for $y \in H_0$, $f^{-1}(y)$ is the intersection of K_∞ with the Hilbert cube

$$\phi^{-1}(y_2) \times I_{2^2} \times I_{2^3} \times \cdots \times \phi^{-1}(y_3) \times I_{3^2} \times \cdots \times \phi^{-1}(y_5) \times I_{5^2} \times \cdots.$$

To show that f is monotone, we must exhibit in $f^{-1}(y)$ a continuum containing any two preassigned points of $f^{-1}(y)$. This proof is entirely analogous to the proof that K_∞ is connected (part (2) of this theorem) and we therefore omit it. Since H_0 is homeomorphic with I^ω, the proof is complete.

COROLLARY. *For each n, there is a monotone map of K_∞ onto an n-cell.*

PROOF. Follow f by the projection of I^ω onto its first n factors. We remark that this corollary is an immediate consequence of Theorem 2; however the proof we give here is somewhat neater in this special case.

5. HID compacta as monotone preimages of arbitrary compacta.

We saw in the previous section that there is an HID space which maps monotonically onto I^ω. The question naturally arises whether, given any compact metric space, there is an HID space which maps monotonically onto it. That the answer is yes is a corollary of Theorem 1, but Theorem 2 gives us a stronger result. We first need the following lemma, which is an extension of Theorem 1:

LEMMA 2. *There is an HID space L and a monotone map $g:L\to I^\omega$ such that for each $p\in I^\omega$, $g^{-1}(p)$ is HID.*

PROOF. Let K_∞ be the HID space of Theorem 1, and regard K_∞ as being embedded in I^ω. Let $C_0\subset K_\infty$ be the Cantor set described in the previous section. Let $I^{\omega'}$ be another Hilbert cube, and let X be a hereditarily indecomposable HID continuum in $I^{\omega'}$ which contains the point $(0, 0, 0, \cdots)$. Then L is the subset of $I^\omega\times I^{\omega'}$ given by $L=(K_\infty\times(0, 0, 0, \cdots))\cup(C_0\times X)$. Intuitively, L is obtained by tacking a copy of X onto K_∞ at each point of C_0. If π is the projection map of $I^\omega\times I^{\omega'}$ onto I^ω, and if f is the monotone map of K_∞ onto I^ω, then the map $g=f\circ\pi|_L$ is a monotone map of L onto I^ω, and it is clear that for each $p\in I^\omega$, $g^{-1}(p)$ is HID. This completes the proof of the lemma.

THEOREM 2. *Let X be a compact metric space. Then there is an HID compactum Y and a monotone map $f\colon Y\to X$ such that for each $p\in X$, $f^{-1}(p)$ is HID. Moreover, the components of Y are in* 1-1 *correspondence with the components of X under the correspondence $c(y)\leftrightarrow c(f(y))$.*

PROOF. Let X be embedded in I^ω, and let g be the monotone map given in Lemma 2. Let $Y=g^{-1}(X)$. Since L is compact and g is a monotone map, the preimage of each component of X is a component of Y. Y is HID since it is an infinite dimensional subspace of an HID space. This completes the proof of the theorem.

COROLLARY. *If X is a separable metric space, all conclusions of Theorem 2 hold except perhaps compactness of Y.*

PROOF. X can be embedded in a compactum $\hat{X}$ of the same dimension [6, Theorem V. 6]. Apply Theorem 2 to $\hat{X}$.

6. **HID compacta as monotone images of finite dimensional compacta.** In §2, we saw that any HID space has a monotone image of any prescribed dimension.

We now observe that any HID space is the monotone image of some space of any prescribed dimension. This is an almost immediate corollary of a result announced by R. D. Anderson in [1], which states that any compact locally connected metric continuum is a monotone-open image of the universal curve under a map f and, moreover, f can be chosen so that each point preimage is homeomorphic to the universal curve. In particular, this is true of the Hilbert cube I^ω; and if X is any compact metric space, which we regard as being embedded in I^ω, then $f|_{f^{-1}(X)}$ is a monotone-open map of a subset of the universal curve onto X. If we let $Y=f^{-1}(X)$, then $Y\times I^{n-1}$ is an n-dimensional space and $f\times\{$projection onto origin$\}$ is a monotone map of $Y\times I^{n-1}\to X$. Moreover, it follows that the preimage of every point under this map has dimension n. For emphasis, we summarize this discussion as follows:

PROPOSITION 2. *Let X be a compact metric space, and let n be any positive integer. Then there is a compact metric space Y of dimension n, and a monotone map $f: Y\to X$. Moreover, $f^{-1}(x)$ has dimension n for each $x\in X$.*

In the case where X is HID, the space Y fails to be locally connected since local connectedness is preserved by monotone mappings and local connectedness together with our other hypotheses would imply arcwise connectedness of X. It would be of interest to have a definitive description of Y in this case; such a description might help in understanding the structure of HID spaces.

It might be remarked that Proposition 2 can also be obtained by modifying the constructions in §§3, 4, and 5 to use n-cells instead of HID continua; however, there is little point in doing that since the proof given here is neater and more intuitively appealing.

7. Questions.

(a) Is every hereditarily indecomposable continuum the monotone image of a hereditarily indecomposable HID continuum?

(b) Is there an HID continuum K'_∞ and a monotone map $f: K'_\infty\to I^\omega$ such that each point preimage is a hereditarily indecomposable HID continuum?

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