SHORTER NOTE

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SOLUTION TO A PROBLEM POSED BY KALICKI

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Kalicki [2] and Łoś [3] independently discovered essentially similar methods for determining effectively whether or not there are any tautologies according to an arbitrary matrix, and Kalicki took certain steps towards obtaining another such test by establishing:

(K) If there exists a sequence of matrices $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ such that $\mathfrak{M}_1$ is cyclic, $\mathfrak{M}_n = \mathfrak{M}$ and for all $m < n \mathfrak{M}_{m+1}$ is either an extension or a repetition of $\mathfrak{M}_m$, then $\mathfrak{M}$ generates an empty set of tautologies.

Kalicki [2] announced verification of the converse of (K) for all matrices with one, two and three elements, but reported that he had not succeeded in proving it generally. (K) together with its converse would of course provide an effective test for the existence of tautologies, but the purpose of this note is to show that no such test emerges. The matrix, $\mathfrak{M}$, set out in truth-table form below (i) generates an empty set of tautologies but (ii) is not obtainable from any cyclic matrix in the manner required by the converse of (K):

For the proof of (i) consider the four matrices of which $\mathfrak{M}$ is an extension:

$$\mathfrak{M} = \begin{array}{c|cccc} \Delta & 1 & 2 & 3 & 4 \\ \hline 1^* & 3 & 1 & 3 & 1 \\ 2^* & 1 & 4 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & 1 & 2 \end{array}$$

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[1] This paper is based on part of Chapter V of the author's Wayne State University doctoral dissertation.

$$\mathfrak{B} = \begin{array}{c|cc} & 1 & 3 \\ \hline 1^* & 3 & 3 \\ 3 & 3 & 1 \end{array} \qquad \mathfrak{C} = \begin{array}{c|cc} & 2 & 4 \\ \hline 2^* & 4 & 2 \\ 4 & 2 & 2 \end{array}$$

$$\mathfrak{D} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1^* & 3 & 1 & 3 & 1 \\ 2^* & 1 & 4 & 1 & 2 \\ 3^* & 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & 1 & 2 \end{array} \qquad \mathfrak{E} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1^* & 3 & 1 & 3 & 1 \\ 2^* & 1 & 4 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 4^* & 1 & 2 & 1 & 2 \end{array}$$

It is known from [1] that every tautology of any matrix is also a tautology of each of the matrices of which it is an extension. Since $\mathfrak{B}$ and $\mathfrak{C}$ have no tautologies in common [1, p. 180], however, it follows that there can be no $\mathfrak{M}$-tautologies.[2]

For the proof of (ii) suppose to the contrary that there is a sequence of matrices, $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$, satisfying the hypothesis of (K). Since $\mathfrak{M} = \mathfrak{M}_n$ is not cyclic, we may assume without loss of generality that $\mathfrak{M}_n \neq \mathfrak{M}_{n-1}$. Now $\mathfrak{M}_n$ cannot be a repetition of any matrix, for there is no way of consistently identifying its designated or undesignated elements. $\mathfrak{M}_n$ must therefore be an extension of $\mathfrak{M}_{n-1}$, and $\mathfrak{M}_{n-1}$ must be one of the matrices $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ or $\mathfrak{E}$. But it is easy to verify that $\Delta p \Delta pp$ is a tautology of $\mathfrak{C}$ and $\mathfrak{D}$, and that $\Delta\Delta p \Delta pp \Delta p \Delta pp$ is a tautology of $\mathfrak{B}$ and $\mathfrak{E}$, so that $\mathfrak{M}_{n-1}$ does not generate an empty set of tautologies. Since $\mathfrak{M}_{n-1}$ satisfies the hypothesis of (K), however, it must generate an empty set of tautologies, and our supposition has been reduced to absurdity.

References

1. J. Kalicki, *Note on truth-tables*, J. Symbolic Logic **15** (1950), 174–181.

2. ———, *A test for the existence of tautologies according to many-valued truth-tables*, J. Symbolic Logic **15** (1950), 182–184.

3. J. Łoś, *On logical matrices*, Trav. Soc. Sci. Lett. Wrocław. Ser. B. no. 19 (1949). (Polish)

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[2] An alternate proof of (i) may be obtained directly from Theorem 1 of [2] or Theorem 25 of [3] by strong induction.