PROOF OF ANDREWS' CONJECTURE ON PARTITION IDENTITIES

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1. **Introduction.** As in [2], we denote by $B_d(n)$ the number of partitions of n of the form $n = b_1 + b_2 + \cdots + b_s$, satisfying the following conditions:

(i) $b_i - b_{i+1} \geqq d$,

(ii) if $d \mid b_i$, then $b_i - b_{i+1} > d$.

We denote by $C_d(n)$ the number of partitions of n satisfying the above conditions and additionally

(iii) $b_s > d$.

These two partition functions appear in several of the well-known identities in the theory of partitions. Thus, the first of the Rogers-Ramanujan identities [5, p. 291] states that $B_1(n)$ is equal to the number of partitions of n into parts $\equiv \pm 1 \pmod 5$, while the second Rogers-Ramanujan identity asserts that $C_1(n)$ is equal to the number of partitions of n into parts $\equiv \pm 2 \pmod 5$. H. Göllnitz [3] and B. Gordon [4] proved independently that $B_2(n)$ is equal to the number of partitions of n into parts $\equiv 1, 4, 7 \pmod 8$, while $C_2(n)$ is the number of partitions of n into parts $\equiv 3, 4, 5 \pmod 8$. I. J. Schur [6] has proved that $B_3(n)$ is equal to the number of partitions of n into parts $\equiv \pm 1 \pmod 6$. For $d > 3$, a theorem by the author [1, p. 713] can easily be extended to prove that $B_d(n)$ is not equal to the number of partitions of n into parts taken from any set of integers whatsoever.

Andrews [2, p. 441] has proved a certain identity involving $C_3(n)$, but states that he has not been able to obtain any simple partition-theoretic interpretation of this identity. He conjectures "that Alder's result for $B_d(n)$ is also valid for $C_d(n)$ with $d > 2$."

It is the object of this paper to prove this conjecture of Andrews. In fact, we prove a more general result, namely that this conjecture is valid if we replace (iii) above by

(iv) $b_s \geqq m$,

where $m \geqq 2$[1], so that Andrews' conjecture is the special case where $m = d + 1$. This result is stated in the following

Received by the editors July 17, 1968 and, in revised form, December 28, 1968.

[1] The author is grateful to the referee for suggesting this even more general inequality than that contained in the author's original manuscript and for proposing a corresponding simplification of the proof.

THEOREM. *The number $C_{d,m}(n)$ of partitions of n into parts of the form $n=b_1+b_2+\cdots+b_s$, with $b_i-b_{i+1}\geqq d$, and if $d|b_i$, then $b_i-b_{i+1}>d$, and $b_s\geqq m$, where $m\geqq 2$, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever if $d>2$.*

As a special case of this theorem it follows that there cannot exist a dual to Schur's Theorem in the sense that the second of the Rogers-Ramanujan identities is a dual to the first one and it also explains why Andrews has not been able to obtain any simple partition-theoretic interpretation of his Theorem 3 in [**2**].

2. **Proof of the theorem.** We suppose that the theorem is false and that there exists such a set of integers $a_1<a_2<a_3<\cdots$; denote this set by A and the number of partitions of n into parts taken from this set by $p_A(n)$. Let n be any integer for which $C_{d,m}(n)\geqq 2$, then $n\geqq(m+d)+m=2m+d$. Hence $C_{d,m}(n)=1$ for $m\leqq n<2m+d$; in particular, $C_{d,m}(2m+2)=1$. Now, since $C_{d,m}(n)=0$ for $1\leqq n<m$, it follows that for $m\geqq 3$, $a_1=m$, $a_2=m+1$, $a_3=m+2$. But then $p_A(2m+2)\geqq 2$, since $2m+2=m+(m+2)=(m+1)+(m+1)$, which is a contradiction. If $m=2$, then $a_1=2$, $a_2=3$, and hence $p_A(6)\geqq 2$, which is again a contradiction.

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