SPLITTING HEREDITARY TORSION THEORIES OVER SEMIPERFECT RINGS

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In this paper we study hereditary torsion theories in the category ${}_R\mathfrak{M}$ of left R-modules over a semiperfect ring R. Specifically we are interested in determining those hereditary torsion theories for which the torsion submodule is a direct summand of every module; we shall use the term *splitting* hereditary torsion theory when this occurs. Our main tool (Theorem 1) is an extension of a theorem of Jans [6]. An important special class of splitting hereditary torsion theories is the class of *centrally splitting* ones; these are determined by central idempotents of the ring. For semiperfect rings in general, it is not known whether every splitting hereditary torsion theory is centrally splitting. However, we show that if R is quasi-Frobenius (QF) then every splitting hereditary torsion theory is centrally splitting. Thus for QF rings there is a one-to-one correspondence between the splitting hereditary torsion theories and the central idempotents in the ring.

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Throughout this paper the term "ring" will mean an associative ring with unit 1, and all modules are assumed to be unitary left modules. For the definition and basic properties of semiperfect rings see [3] or [7]. Dickson [4] defined a *torsion theory* for ${}_R\mathfrak{M}$ to be a pair $(\mathfrak{I}, \mathfrak{F})$ of classes of modules satisfying:

(a) $\mathfrak{I} \cap \mathfrak{F} = 0$;

(b) $\mathfrak{I}$ is closed under homomorphic images and $\mathfrak{F}$ is closed under submodules;

(c) for each module M there exists a (unique) submodule M_t of M such that $M_t \in \mathfrak{I}$ and $M/M_t \in \mathfrak{F}$.

A class of modules $\mathfrak{I}$ is a *torsion class* if there exists a class $\mathfrak{F}$ such that $(\mathfrak{I}, \mathfrak{F})$ is a torsion theory; *torsion-free class* is defined dually. A torsion class $\mathfrak{I}$, and the associated torsion theory $(\mathfrak{I}, \mathfrak{F})$, is called *hereditary* if $M \in \mathfrak{I}$ and $0 \to N \to M$ exact implies that $N \in \mathfrak{I}$.

The following facts occur in [4]:

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(A) A class $\mathfrak{T}$ of modules is a torsion class if and only if $\mathfrak{T}$ is closed under homomorphic images, arbitrary direct sums, and extensions. Dually, a class $\mathfrak{F}$ is a torsion-free class if and only if $\mathfrak{F}$ is closed under submodules, arbitrary direct products, and extensions.

(B) Let $(\mathfrak{T}, \mathfrak{F})$ be a torsion theory; then $\mathfrak{T}$ and $\mathfrak{F}$ uniquely determine each other. Specifically,

$$\mathfrak{T} = \{M \mid \operatorname{Hom}(M, F) = 0 \quad \text{for all } F \in \mathfrak{F}\}, \quad \text{and}$$
$$\mathfrak{F} = \{N \mid \operatorname{Hom}(T, N) = 0 \quad \text{for all } T \in \mathfrak{T}\}.$$

(C) If $(\mathfrak{T}, \mathfrak{F})$ is a torsion theory, then $\mathfrak{T}$ is hereditary if and only if $\mathfrak{F}$ is closed under injective envelopes.

A hereditary torsion class $\mathfrak{T}$ is a strongly complete Serre class, and Gabriel [**5**] has shown that there is a one-to-one correspondence between such classes and strongly complete filters $F(\mathfrak{T})$ of left ideals of the ring R, where $F(\mathfrak{T})$ denotes the set of all left ideals I of R such that $R/I \in \mathfrak{T}$. We call $F(\mathfrak{T})$ the *torsion filter* of $\mathfrak{T}$.

(D) Let $(\mathfrak{T}, \mathfrak{F})$ be a hereditary torsion theory. For each module M we have that

$$\begin{aligned} M_t &= \sum \{T \subseteq M \mid T \in \mathfrak{T}\} \\ &= \bigcap \{K \subseteq M \mid M/K \in \mathfrak{F}\} \\ &= \{x \in M \mid (0:x) \in F(\mathfrak{T})\} \end{aligned}$$

where $(0:x) = \{r \in R \mid rx = 0\}$.

In [**6**] Jans defined a hereditary torsion class $\mathfrak{T}$ to be a *torsion-torsion-free* (TTF) class provided $\mathfrak{T}$ is closed under direct products. If $(\mathfrak{T}, \mathfrak{F})$ is a hereditary torsion theory and if $\mathfrak{T}$ is a TTF class, then $\mathfrak{T}$ is also a torsion-free class for some torsion class $\mathfrak{C}$ by (A). We call $(\mathfrak{T}, \mathfrak{F})$ and $(\mathfrak{C}, \mathfrak{T})$ the torsion theories *associated* with $\mathfrak{T}$; by (C) $\mathfrak{C}$ is hereditary if and only if $\mathfrak{T}$ is closed under injective envelopes. By a theorem of Pierce in [**6**], $\mathfrak{T}$ is a TTF class if and only if the torsion filter $F(\mathfrak{T})$ has a unique minimal left ideal, and Jans has shown that this (necessarily idempotent) ideal is R_c, the $\mathfrak{C}$-torsion submodule of R. Furthermore, R_t and R_c are two-sided ideals.

THEOREM 1. *Let R be an arbitrary ring, let $\mathfrak{T}$ be a TTF class, and let $(\mathfrak{T}, \mathfrak{F})$ and $(\mathfrak{C}, \mathfrak{T})$ be the torsion theories associated with $\mathfrak{T}$. The following are equivalent*:

(a) $R = R_t \oplus R_c$ *(ring direct sum)*;

(b) $M = M_t \oplus M_c$ *for all modules M*;

(c) $\mathfrak{F} = \mathfrak{C}$;

(d) $(M_c)_t = 0$ and $(M/M_t)_c = M/M_t$ *for all modules M*;

(e) $\mathfrak{I}$ *is closed under injective envelopes and* R_c *is a direct summand of* R;

(f) $\mathfrak{F}$ *is closed under homomorphic images and* R_t *is a direct summand of* R;

(g) R_c *is a ring direct summand of* R.

PROOF. The equivalence of (a), (b), (c), and (d) is by Jans [**6**, Theorem 2.4]. Clearly (a) and (c) imply (e), (f), and (g).

(e)$\Rightarrow$(a). By (C), $\mathfrak{I}$ is closed under injective envelopes if and only if $\mathfrak{C}$ is hereditary. Let $R=R_c\oplus I'$; then $I'\cong R/R_c\in\mathfrak{I}$ and thus $I'\subseteq R_t$. Hence $R=R_c+R_t$. But $\mathfrak{C}$ and $\mathfrak{I}$ are each hereditary, so $R_c\cap R_t\in\mathfrak{C}\cap\mathfrak{I}=0$.

(f)$\Rightarrow$(a). Let $R=R_t\oplus I''$; then $R/R_t\cong I''\in\mathfrak{F}$. Also we have $R_t\cong R/I''\in\mathfrak{I}$, so that $I''\in F(\mathfrak{I})$ and $R_c\subseteq I''$. But note that $I''\rightarrow I''/R_c\rightarrow 0$ is exact and that $0\rightarrow I''/R_c\rightarrow R/R_c$ is exact, so that $I''/R_c\in\mathfrak{I}\cap\mathfrak{F}=0$. Thus $I''=R_c$.

(g)$\Rightarrow$(a). Let $R=R_c\oplus I'''$ (ring direct sum); then $I'''\cong R/R_c\in\mathfrak{I}$ so that $I'''\subseteq R_t$ and $R=R_c+R_t$. Also $R_c=R\epsilon$ where ϵ is a central idempotent; thus if $x=a\epsilon\in R_c\cap R_t$, then by (D) we have $(0:x)\in F(\mathfrak{I})$ and $R_cx=0$. Hence $0=\epsilon x=x$.■

We call $(\mathfrak{I}, \mathfrak{F})$ *centrally splitting* provided $\mathfrak{I}$ is a TTF class with associated torsion theories $(\mathfrak{I}, \mathfrak{F})$ and $(\mathfrak{C}, \mathfrak{I})$ satisfying (a)–(g) of Theorem 1. This name was suggested by the next result.

PROPOSITION 2. *If* R *is an arbitrary ring, then* $(\mathfrak{I}, \mathfrak{F})$ *is centrally splitting if and only if* $\mathfrak{I}=\mathfrak{I}_\epsilon$ *and* $\mathfrak{F}=\mathfrak{F}_\epsilon$ *for some central idempotent* ϵ *of* R, *where*

$$\mathfrak{I}_\epsilon = \{M \mid (1-\epsilon)M = M\} \quad \textit{and} \quad \mathfrak{F}_\epsilon = \{N \mid \epsilon N = N\}.$$

PROOF. That $(\mathfrak{I}_\epsilon, \mathfrak{F}_\epsilon)$ satisfies (f) of Theorem 1, and that $\mathfrak{I}_\epsilon$ is closed under direct products, is easily seen. Thus $(\mathfrak{I}, \mathfrak{F})=(\mathfrak{I}_\epsilon, \mathfrak{F}_\epsilon)$ implies centrally splitting.

Conversely, assume that $(\mathfrak{I}, \mathfrak{F})$ is centrally splitting. Then, by (a) of Theorem 1, there exists a central idempotent ϵ of R such that $R_c=R\epsilon$ and $R_t=R(1-\epsilon)$. Using the facts that $\mathfrak{I}$ and $\mathfrak{F}$ are each closed under homomorphic images and that $M=\epsilon M\oplus(1-\epsilon)M$ for any module M, the result can now be checked.■

It is not difficult to see that $\epsilon\leftrightarrow(\mathfrak{I}_\epsilon, \mathfrak{F}_\epsilon)$ gives a one-to-one correspondence between the centrally splitting torsion theories and the central idempotents of R.

Let $(\mathfrak{I}, \mathfrak{F})$ be a hereditary torsion theory; if this torsion theory is splitting, then necessarily every indecomposable module is either torsion or torsion-free. In particular, for such a torsion theory every principal indecomposable module is either torsion or torsion-free (a module is *principal indecomposable* in case it is isomorphic to Re for some primitive idempotent e in R). Moreover, if R is a semiperfect ring and if e is a primitive idempotent in R, then Je is the (unique) largest submodule of Re, where J denotes the Jacobson radical of R. Thus Re/Je is simple, and so it is either in $\mathfrak{J}$ or in $\mathfrak{F}$.

We shall call $(\mathfrak{J}, \mathfrak{F})$ *principal* provided

(i) $Re \in \mathfrak{J}$ if and only if $Re/Je \in \mathfrak{J}$, and

(ii) $Rf \in \mathfrak{F}$ if and only if $Rf/Jf \in \mathfrak{F}$,

for all principal indecomposable modules Re and Rf.

Recall that over a semiperfect ring the Jacobson radical of a module M is JM.

PROPOSITION 3. *Let R be a semiperfect ring, and let $(\mathfrak{J}, \mathfrak{F})$ be a principal hereditary torsion theory. If M is a finitely generated module and if P is the projective cover* (*see* [3]) *of M, then the following are equivalent*:

(a) $M \in \mathfrak{J}$;

(b) $M/JM \in \mathfrak{J}$;

(c) $P \in \mathfrak{J}$.

Moreover, we can replace $\mathfrak{J}$ by $\mathfrak{F}$ in (a), (b), *and* (c) *above.*

PROOF. Trivially $P \in \mathfrak{J}$ implies $M \in \mathfrak{J}$ and $M \in \mathfrak{J}$ implies $M/JM \in \mathfrak{J}$; thus assume that $M/JM \in \mathfrak{J}$. Then

$$M/JM \cong \oplus \sum \{Re_\alpha/Je_\alpha \mid \alpha \in A\} \quad \text{and}$$
$$P \cong \oplus \sum \{Re_\alpha \mid \alpha \in A\}$$

where e_α is a primitive idempotent for each $\alpha \in A$ (see [7, Exercise 15, p. 94]). Hence $Re_\alpha/Je_\alpha \in \mathfrak{J}$ for each $\alpha \in A$, so that $P \in \mathfrak{J}$ since $(\mathfrak{J}, \mathfrak{F})$ is principal and $\mathfrak{J}$ is closed under direct sums.

To get the corresponding results for $\mathfrak{F}$, choose $F \in \mathfrak{F}$ and assume that $F \to N \to 0$ is exact. If $x \in N_t$, then we have just seen that the projective cover Q of Rx belongs to $\mathfrak{J}$. Thus we have the diagram

$$\begin{array}{ccccc} & & Q & & \\ & & \downarrow & & \\ & & Rx & & \\ & & \downarrow & & \\ F & \to & N & \to & 0. \end{array}$$

Since Q is projective, there exists a homomorphism from Q to F such that $Q \to F \to N = Q \to Rx \to N$. But $\mathrm{Hom}(Q, F) = 0$ by (B), so that $x = 0$ and hence $N_t = 0$. We have shown that $\mathfrak{F}$ is closed under homomorphic images; the concluding statement follows now by the previous proof for $\mathfrak{I}$.■

COROLLARY 4. *Let R be a semiperfect ring, and let $(\mathfrak{I}, \mathfrak{F})$ be a principal hereditary torsion theory. Then $\mathfrak{F}$ is closed under homomorphic images.*

THEOREM 5. *Let R be a semiperfect ring, and let $(\mathfrak{I}, \mathfrak{F})$ be a hereditary torsion theory. If $\mathfrak{I}$ is a TTF class, then $(\mathfrak{I}, \mathfrak{F})$ is principal if and only if $(\mathfrak{I}, \mathfrak{F})$ is centrally splitting.*

PROOF. Assume first that $(\mathfrak{I}, \mathfrak{F})$ is principal; let

$$I = \sum \{Re \subseteq R \mid e \text{ is a primitive idempotent and } Re \in \mathfrak{I}\}$$

and

$$I' = \sum \{Rf \subseteq R \mid f \text{ is a primitive idempotent and } Rf \in \mathfrak{F}\}.$$

Since R is a semiperfect ring, $R = I \oplus I'$. Also $I \subseteq R_t$ clearly, and $R/I \cong I' \in \mathfrak{F}$ so that $R_t \subseteq I$ by (D). Thus $R_t = I$. But $\mathfrak{I}$ is a TTF class, and by Corollary 4 $\mathfrak{F}$ is closed under homomorphic images. Hence (f) of Theorem 1 holds, and so $(\mathfrak{I}, \mathfrak{F})$ is centrally splitting.

Assume that $(\mathfrak{I}, \mathfrak{F})$ is centrally splitting with $\mathfrak{I} = \mathfrak{I}_\epsilon$ and $\mathfrak{F} = \mathfrak{F}_\epsilon$ for a central idempotent ϵ of R. It is easy to see that $\mathfrak{F}$ is closed under homomorphic images, so that $Re \in \mathfrak{I}(\mathfrak{F})$ implies that $Re/Je \in \mathfrak{I}(\mathfrak{F})$ trivially. Suppose that $Re/Je \in \mathfrak{F}$; then $\epsilon(Re/Je) = Re/Je$ and hence $\epsilon(Re) = Re$ since Je is the (unique) largest submodule of Re. Thus $Re \in \mathfrak{F}$; the corresponding proof for $\mathfrak{I}$ is nearly identical.■

If R is a right perfect ring, then Alin [1, Corollary 2.3.3] has shown that every hereditary torsion class is a TTF class. Thus for a right perfect ring, principal is equivalent to centrally splitting.

If $(\mathfrak{I}, \mathfrak{F})$ is a torsion theory for ${}_R\mathfrak{M}$, then every simple module belongs either to $\mathfrak{I}$ or to $\mathfrak{F}$; thus the simple modules are partitioned into two disjoint classes. On the other hand, if R is semisimple then every module is a direct sum of simple modules. Hence any partition of the simple modules into two disjoint classes closed under isomorphisms characterizes a torsion theory for ${}_R\mathfrak{M}$.

COROLLARY 6. *If R is semisimple, then every torsion theory for ${}_R\mathfrak{M}$ is centrally splitting.*

PROOF. By Theorem 5, since every torsion theory is trivially principal.■

From Corollary 6 (or directly) we infer that every torsion theory for ${}_R\mathfrak{M}$ is hereditary when R is semisimple. This assertion can also be shown to be true if R is a local uniserial ring, since then the only torsion theories for ${}_R\mathfrak{M}$ are the trivial ones.

Now we turn to QF rings; for this case we can characterize all of the splitting hereditary torsion theories.

THEOREM 7. *Let R be a QF ring and let $(\mathfrak{I}, \mathfrak{F})$ be a hereditary torsion theory for ${}_R\mathfrak{M}$. Then the following are equivalent*:

(a) $\mathfrak{I}$ *is closed under injective envelopes*;

(b) $(\mathfrak{I}, \mathfrak{F})$ *is principal*;

(c) $(\mathfrak{I}, \mathfrak{F})$ *is centrally splitting*;

(d) $(\mathfrak{I}, \mathfrak{F})$ *is splitting.*

PROOF. That (b)⇒(c) follows from Theorem 5, and (c)⇒(d) is trivial.

(a)⇒(b). Write R as a finite direct sum of principal indecomposable modules, and suppose that $\{Re_1, \cdots, Re_n\}$ is a basic set for R; i.e., every principal indecomposable module is isomorphic to one and only one element of this set. After reindexing, if necessary, we may assume that

$$\{Re_1/Je_1, \cdots, Re_k/Je_k\} \subset \mathfrak{I} \text{ and}$$
$$\{Re_{k+1}/Je_{k+1}, \cdots, Re_n/Je_n\} \subset \mathfrak{F}.$$

If $Re_i \in \mathfrak{I}$, then clearly $Re_i/Je_i \in \mathfrak{I}$. On the other hand, suppose that $Re_i/Je_i \in \mathfrak{I}$. Then, denoting the injective envelope of a module M by $E(M)$, we have $E(Re_i/Je_i) \cong Re_{\delta(i)} \in \mathfrak{I}$, where δ is a permutation on $\{1, \cdots, n\}$ (see [8]), so that $Re_{\delta(i)}/Je_{\delta(i)} \in \mathfrak{I}$ and $\delta(i) \in \{1, \cdots, k\}$. Thus δ is a permutation defined on $\{1, \cdots, k\}$, and thus $Re_i \cong Re_{\delta(v)}$ for some $v \in \{1, \cdots, k\}$. Hence $Re_i \in \mathfrak{I}$.

Now assume that $Re_j \in \mathfrak{F}$; then Re_j/Je_j is simple, so that either Re_j/Je_j is in $\mathfrak{I}$ or in $\mathfrak{F}$. But the preceding argument shows that Re_j/Je_j cannot be in $\mathfrak{I}$; thus $Re_j/Je_j \in \mathfrak{F}$.

The argument that $Re_j/Je_j \in \mathfrak{F}$ implies $Re_j \in \mathfrak{F}$ is now identical to the corresponding proof for $\mathfrak{I}$.

(d)⇒(a). Choose $T \in \mathfrak{I}$ and let $E = E(T)$. Then $E = E_t \oplus E'$ and $T \subseteq E_t$. But T is essential in E, so that $E' = 0$. ∎

We remark that the standard torsion class over an integral domain is always closed under injective envelopes, as is the Goldie torsion class over any ring (see [1, p. 14]). Thus Theorem 7 gives a generalization of some results of Alin and Dickson [2], who show that the Goldie torsion theory splits when the ring is QF. Both the simple torsion class (see [1]) and the $E(R)$-torsion class studied by Jans in [6] are trivial for QF rings. However, Kent R. Fuller has pointed out to the author that the ring of upper triangular matrices over a field of the form

$$\begin{bmatrix} b & x & y \\ 0 & a & z \\ 0 & 0 & b \end{bmatrix}$$

provides an example which shows that the $E(R)$-torsion class is not closed under injective envelopes for generalized uniserial rings.

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