COMMUTATIVE SEMIGROUP LAWS[1]

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1. **Introduction.** It is a consequence of B. H. Neumann's classification of group identities [**3**, Theorem 19.1, p. 523], that the lattice of Abelian group varieties is distributive. The lattice of varieties of algebras in one unary operation is also distributive [**2**], but the lattice of commutative semigroup varieties is not modular [**4**]. Here we discuss a distributive sublattice of this nonmodular lattice.

By variety we will mean commutative semigroup variety. (Lemmas 1 and 2, however, are true for semigroup varieties, not necessarily commutative.) The semigroups need not have a unit-element. We will be mainly concerned with laws of the form $s=sx^a$ where s is a term (word in the variables), x is variable and a is a positive integer. We call such a law an L-law and call a variety which can be defined by a set $\{s_i=s_ix^{a_i}\}$ of L-laws, an L-variety.

2. L-**Laws.** Exponents of variables will always be positive integers. Lemma 1 is easily proved by induction on k, where $b=ka$.

LEMMA 1. *Let s be a term and x be a variable. If b is a multiple of a, then $s=sx^b$ holds in the variety defined by $s=sx^a$.*

LEMMA 2. *Let s, t be terms and x be a variable. If d is the greatest common divisor of a and b, then $s=sx^d$ holds in the variety defined by $s=sx^a$ and $t=tx^b$.*

PROOF. The substitution of x for each variable in $t=tx^b$ yields $x^p=x^{p+b}$, where p is some positive integer. Hence we have $sx^p=sx^{p+b}$, and thus, by Lemma 1, $sx^p=sx^{p+jb}$, $j=1, 2, \cdots$.

From $s=sx^a$ we obtain $sx^p=sx^{p+ia}$, $i=1, 2, \cdots$. Hence $sx^p=sx^{p+ia+jb}$, $i, j=0, 1, 2, \cdots$.

Thus, by an elementary property of nonnegative integers, $sx^p=sx^{p+ka+d}$ for some nonnegative integers k.

From this last law and $s=sx^{ka}$, we obtain $sx^p=sx^{p+d}$. So $sx^{ra}=sx^{ra}x^d$, where $ra\geqq p$. Hence, since $s=sx^{ra}$, we have $s=sx^d$.

By $n(x, s)$ we mean the number ($\geqq 0$) of occurrences of a variable x in a term s.

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LEMMA 3. *If $n(x, s) \geqq n(y, s)$, then $s = sx^a$ holds in the variety defined by $s = sy^a$.*

PROOF. Suppose (i, j), where i, j are positive integers, denotes the term $x^i y^j$. Then each of the following laws (after the first) can be obtained from the previous one.

$$\begin{aligned}
(p, q) &= (p, q + a), \\
(q + ka, q) &= (q + ka, q + a), \quad \text{if } q + ka \geqq p, \\
(q + ka, q) &= (q + ka, q + ka), \\
(q + ka, q) &= (q, q + ka), \\
(p + ka, q) &= (p, q + ka), \quad \text{if } p \geqq q.
\end{aligned}$$

Similarly, from $s = sy^a$ we can derive $sx^{ka} = sy^{ka}$, where k is a positive integer such that $n(y, s) + ka \geqq n(x, s)$.

By a familiar argument, the last law, with $s = sy^a$, yields $s = sx^{ka}$.

From $s = sy^a$, by substitution, we have $t = tx^a$ for some term t. Hence, since a is the g.c.d. of ka and a, we have, by Lemma 2, $s = sx^a$.

We define a simple closure operation on the set of terms as follows:

(i) The closure $\text{cl}(S)$ of a set S of terms is the union of the closures of the one element subsets of S.

(ii) For terms s, t, $t \in \text{cl}(\{s\}) = \text{cl}(s)$, in case there is a function from the set of variables of s into the set of variables of t such that if distinct variables $x_1, \cdots, x_m$ are mapped to a variable x, then

$$n(x_1, s) + \cdots + n(x_m, s) \leqq n(x, t).$$

Thus for variables x, y and any term t, x^3y^4t and x^7t are in $\text{cl}(x^3y^4)$. For any term t denote by $\text{Sg}(t)$ the free commutative semigroup generated by the variables occurring in t. Then (ii) says: There is a homomorphism $\phi: \text{Sg}(s) \to \text{Sg}(t)$, that maps variables into variables, such that t lies in the ideal generated by $\phi(s)$.

LEMMA 4. *If $n(x, t) \geqq n(x, s_1)$ and $t \in \text{cl}(s_2)$ then $t = tx^a$ holds in the variety defined by the laws $s_1 = s_1x^a$, $s_2 = s_2x^a$.*

PROOF. It is readily seen that from $s_1 = s_1x^a$ we can obtain a law $s = sy^a$, where y is a variable not in t, s is a term that does not contain x or any of the variables of t, and $n(x, s_1) = n(y, s)$. Then from $s = sy^a$, we have $ts = tsy^a$. Also

$$n(x, ts) = n(x, t) \geqq n(x, s_1) = n(y, s) = n(y, ts).$$

Thus by Lemma 3, we have $ts = tsx^a$.

Since $t \in \text{cl}(s_2)$, we find, using $s_2 = s_2x^a$, that $t = tu^a$ for some variable u. The substitution of u^a for each variable of s in $ts = tsx^a$ leads to

$tu^{ka} = tu^{ka}x^a$ for some positive integer k. This last law, with $t = tu^a$, yields $t = tx^a$.

LEMMA 5. *Let* $E = \{s_i = s_i x^{a_i}\}$ *be a set of L-laws. Then the L-law* $t = tx^b$ *holds in the variety defined by E if and only if*

(i) b *is a multiple of* g.c.d. $\{a_i\}$,

(ii) $n(x, t) \geqq \min\{n(x, s_i)\}$ *and,*

(iii) $t \in \text{cl}(\{s_i\})$.

PROOF. Let τ denote the law $t = tx^a$. Suppose τ holds in the variety V defined by E. The cyclic group of order g.c.d. $\{a_i\}$ is (as a semigroup) an algebra of V, hence τ must satisfy condition (i).

Let $p = \min\{n(x, s_i)\}$ and suppose $p > 0$ (otherwise (ii) is trivial). Let A be the commutative semigroup with two generators a, b defined by $a^p = a^{p+1}$, $b^p = b^{p+1}$. Clearly A is in V and if $n(x, t) < p$, τ cannot hold in A. (In τ substitute a for x and b for the other variables, if any, of τ.)

Define an equivalence relation R on the set of terms by sRt if and only if $s = t$ holds in every commutative semigroup. Let $[s]$ denote the equivalence class containing s and define a binary operation on the set Q of equivalence classes by $[s][t] = [st]$. Let P be the set of all $[s]$ such that $s \in \text{cl}(\{s_i\})$. Then P is an ideal of Q. Let $B = Q/P$ be the Rees factor semigroup of Q modulo P [**1**, p. 17]. Then B is in V and for τ to hold in B, τ must satisfy condition (iii).

Conversely suppose conditions (i)–(iii) are satisfied. Then by Lemmas 1, 2, and 4, τ holds in the variety defined by E.

3. The lattice of L-varieties. It follows from Lemma 5 that if two sets $\{s_i = s_i x^{a_i}\}$ and $\{t_i = t_i x^{b_i}\}$ of L-laws define the same variety then

(i) g.c.d. $\{a_i\}$ = g.c.d. $\{b_i\}$,

(ii) $\min\{n(x, s_i)\} = \min\{n(x, t_i)\}$, and

(iii) $\text{cl}(\{s_i\}) = \text{cl}(\{t_i\})$.

Thus if V is the variety defined by a set $\{s_i = s_i x^{a_i}\}$ we let

(i) period $V \equiv$ g.c.d. $\{a_i\}$,

(ii) level $V \equiv \min\{n(x, s_i)\}$, and

(iii) scope $V \equiv \text{cl}(\{s_i\})$.

Let ϕ be a law $s = t$ with $a = n(x, s)$ and $b = n(x, t)$. We make the following definitions

(i) Period of x in $\phi \equiv |a - b|$.

Period $\phi \equiv$ g.c.d. of the periods of the variables of ϕ.

(ii) Level of x in $\phi \equiv \min(a, b)$.

Level of $\phi \equiv$ minimum of the levels of the variables of ϕ with nonzero periods.

(iii) Scope $\phi \equiv \mathrm{cl}(\{s, t\})$.
(Thus the period, level and scope respectively of $s = sx^a$ is a, $n(x, s)$, and $\mathrm{cl}(s)$.)

We say a law ϕ is trivial in case ϕ holds in every commutative semigroup.

THEOREM 1. *Let V be an L-variety. A nontrivial law ϕ holds in V if and only if*

(i) period ϕ *is a multiple of* period V

(ii) level $\phi \geqq$ level V *and*

(iii) scope $\phi \subseteq$ scope V.

PROOF. We omit the proof of the necessity, since it is similar to the proof of the necessity in Lemma 5.

Suppose $\phi: s = t$ satisfies conditions (i), (ii), and (iii). Let $x_1, \cdots, x_m$ $(y_1, \cdots, y_n)$ be the variables that appear more often in $s(t)$ than in $t(s)$ and let $c_i(d_i)$ be the period in $s = t$ of $x_i(y_i)$. By Lemma 5 we have that $s = sy_1^{d_1}, \cdots, s = sy_n^{d_n}$, $t = tx_1^{c_1}, \cdots, t = tx_m^{c_m}$ hold in V. Hence $s = sy_1^{d_1} \cdots y_n^{d_n}$ and $t = tx_1^{c_1} \cdots x_m^{c_m}$ hold in V. From these two laws and the trivial law $sy_1^{d_1} \cdots y_n^{d_n} = tx_1^{c_1} \cdots x_m^{c_m}$, we have that ϕ holds in V.

LEMMA 6. *Given two L-varieties with levels p, q and* scope A, B *respectively there is an L-variety with* level $\max(p, q)$, scope $(A \cap B)$, *and* period a *for any* $a > 0$.

PROOF. Let $C = \{t \in A \cap B \mid n(x, t) \geqq \max(p, q)\}$, $E = \{t = tx^a \mid t \in C\}$, then the variety defined by E has the desired properties.

We denote the join of two varieties V, W by $V + W$. The next theorem follows from Theorem 1, Lemma 6, and an observation dual to Lemma 6, involving min instead of max and $\cup$ instead of $\cap$.

THEOREM 2. *Let V and W be L-varieties. Then $V \cap W$ and $V + W$ are L-varieties and the mapping*

$$V \rightarrow (\text{period } V, \text{level } V, \text{scope } V)$$

between the lattice of L-varieties under $\cap$ and $+$ and the direct product of

(i) *the lattice of positive integers under* g.c.d. *and* l.c.m.,

(ii) *the lattice of nonnegative integers under* min *and* max, *and*

(iii) *the dual of the lattice of all closed sets of terms under $\cap$ and $\cup$ is an injective homomorphism.*

The next theorem follows immediately from Theorem 2.

THEOREM 3. *The L-varieties form a distributive sublattice of the lattice of commutative semigroup varieties.*

The mapping considered in Theorem 2 is not an isomorphism. For example, there is no L-variety with scope equal to the set of all terms and level $\geqq 2$.

4. On other laws. Since the lattice of varieties is not modular, not every variety is an L-variety. A simple example of a non-L-variety is the variety defined by $xy^2 = x^2y$; other examples occur in [**4**]. We consider two types of laws that define L-varieties.

THEOREM 4. *Let s, t, be terms. The variety defined by $s = st$ is an L-variety.*

PROOF. We show $\phi: s = st$ is equivalent to the L-law $\sigma: s = sx^a$ where $a =$ period ϕ and $n(x, s) =$ level ϕ. (Thus x is a variable of t such that $n(x, s) =$ level ϕ.) By Theorem 1 from σ, we have ϕ.

Conversely the substitution of x for each variable of ϕ and the substitution of x^2 for x and x for the other variables, if any, of ϕ yields laws in x of periods $q+b$ and $q+2b$ respectively, where q is some nonnegative integer and b is the period of x in ϕ. Thus, since g.c.d. $(q+b, q+2b)$ divides b, we have $x^p = x^{p+b}$, where p is some positive integer. Similarly we can obtain L-laws of the periods of the other variables of ϕ in t, so we have an L-law of period $\phi =$ period σ.

From $s = st$ and $x^p = x^{p+a}$ respectively, we have $s = st^k$ and $x^{ka} = x^{2ka}$, where $ka \geqq p$. These last two laws lead to $s = sx^{ka}$, an L-law of level σ and scope σ. Thus we have σ.

THEOREM 5. *If some variable occurs in only one of the terms, s, t then the variety defined by $s = t$ is an L-variety.*

PROOF. Suppose $n(x, s) = 0$ and $n(x, t) = p > 0$. By the previous theorem, it suffices to show that $s = t$ is equivalent to the two laws $s = s^p t$, $t = s^p t$. It is obvious that these laws imply $s = t$. On the other hand the substitution of sx for x in $s = t$ leads to $s = s^p t$, which with $s = t$, yields $t = s^p t$.

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