

# BOUNDED AND CONTINUOUS RANDOM FOURIER SERIES ON NONCOMMUTATIVE GROUPS

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1. **Introduction.** The following two results were proved by P. Billard:

I. Let  $\xi_n$  ( $n = \pm 1, \pm 2, \dots$ ) be independent random variables equidistributed on the interval  $(0, 2\pi]$ . Consider the random Fourier series

$$(1) \quad \sum_{n=-\infty}^{+\infty} a_n \exp[in(x + \xi_n)],$$

where  $a_n$  are fixed complex numbers. Then if (1) represents with probability one a function in  $L^\infty(0, 2\pi)$ , it represents with probability one a continuous periodic function [1, Proposition A].

II. Let  $\epsilon_n$  be independent random variables taking the values  $+1$  and  $-1$  with equal probability. Consider the random Fourier series

$$(2) \quad \sum_{n=-\infty}^{+\infty} \epsilon_n a_n \exp(inx),$$

where  $a_n$  are fixed complex numbers. Then if (2) represents with probability one a function in  $L^\infty(0, 2\pi)$ , it represents with probability one a continuous periodic function [2, Théorème 1].

The purpose of this note is to show that while I holds, when appropriately interpreted, for Fourier series on arbitrary compact groups, II does not hold in general if the group is noncommutative.

The fact that a generalization of I holds for arbitrary groups follows easily from Billard's original proof and some results of Kahane on random series in Banach spaces [4, Théorème 2] and [5, Chapter II, Theorem 1]. With the help of another result of Kahane [4, Théorème 3] and [5, Chapter II, Theorem 5] one can easily show that II holds for Fourier series defined on an arbitrary commutative compact group. Our example in §3 shows that II can fail at least for some noncommutative groups.

One should notice that II is a stronger result than I, in fact an application of Fubini's theorem yields that if II is true, the conclusion also holds for random Fourier series of the type

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$$(3) \quad \sum_{n=-\infty}^{+\infty} X_n \exp(inx),$$

where  $X_n$  are independent symmetric complex valued random variables (symmetric means that  $X_n$  and  $-X_n$  are similar) [2, Théorème 6].

**2. Billard's theorem for compact groups.** Let  $G$  be a compact group. Let  $\{D_n\}_{n=1}^{\infty}$  be a sequence of nonequivalent irreducible unitary representations of  $G$  and let  $d_n$  be the degree of  $D_n$ . Let  $\xi_n$  be a sequence of independent random variables with values on  $G$ . We assume that  $\xi_n$  are *equidistributed*, i.e. that the measure induced by each of them on  $G$  is the Haar measure. Let  $\epsilon_n$  be independent random variables, also independent of the  $\xi_n$ , taking with equal probability, the values  $+1$  and  $-1$ . We shall call the  $\epsilon_n$  the Bernoulli random variables. Let  $A_n$  be fixed  $d_n \times d_n$  matrices. We consider the random Fourier series

$$(4) \quad \sum_{n=1}^{\infty} \epsilon_n d_n \operatorname{tr}(A_n D_n(\xi_n x)).$$

We want to prove the following

**THEOREM.** *If (4) represents with probability one a function in  $L^\infty(G)$ , then it represents with probability one a continuous function.*

For the proof we use the following result of Kahane [4, Théorème 2] and [5, Chapter II, Theorem 1].

**LEMMA 1.** *Let  $X_n$  be independent random variables with values in a Banach space. Suppose that for each  $n$ ,  $-X_n$  and  $X_n$  are similar (i.e. they have the same probability distribution). Let  $S = \{a_{nm}\}$  be an infinite matrix satisfying  $\lim_n a_{nm} = 1$ . Then if the event defined by the relations*

$$\limsup_{p, p' \rightarrow \infty} \left\| \sum_{m=p}^{p'} a_{nm} X_m \right\| = 0 \quad (n = 1, 2, \dots)$$

$$\limsup_{n \rightarrow \infty} \limsup_{p \rightarrow \infty} \left\| \sum_{m=1}^p a_{nm} X_m \right\| < \infty,$$

*has probability one, the event defined by the relation*

$$\limsup_{p \rightarrow \infty} \left\| \sum_{m=1}^p X_m \right\| < \infty$$

*also has probability one.*

COROLLARY. If (4) represents with probability one a function in  $L^\infty$ , then the event

$$\limsup_{p \rightarrow \infty} \left\| \sum_{n=1}^p \epsilon_n d_n \operatorname{tr}(A_n D_n(\xi_n x)) \right\|_\infty < \infty$$

has probability one.

PROOF. Let  $k_\alpha$  be a central approximate identity of  $L^1(G)$  consisting of trigonometric polynomials. That is (i)  $k_\alpha$  has a finite Fourier series development, (ii)  $\|k_\alpha\|_1 \leq 1$  and  $\lim_\alpha \|k_\alpha * f - f\|_1 = 0$ , for every  $f \in L^1$ . Since the closed ideal of  $L^1$  generated by the characters of the representations  $\{D_n\}_{n=1}^\infty$  is separable, we can choose a sequence  $\{k_n\} \subseteq \{k_\alpha\}$  such that  $\lim_n \|k_n * f - f\|_1 = 0$  for every  $f$  belonging to this ideal. Let  $e_n$  be the character of  $D_n$  and let  $k_n * e_m = a_{nm} e_m$ . Then  $\lim_n a_{nm} = 1$ . If  $f_\omega$  ( $\omega \in \Omega$ , the probability space on which the random variables are defined) is the  $L^\infty$  function represented by (4), then

$$(5) \quad k_n * f_\omega = \sum_{m=1}^\infty \epsilon_m a_{nm} d_m \operatorname{tr}(A_m D_m(\xi_m x)),$$

where all but finitely many coefficients  $a_{nm}$  are zero. Thus  $k_n * f_\omega \in C(G)$  and  $\|k_n * f_\omega\|_\infty \leq \|k_n\|_1 \|f_\omega\|_\infty$ ; therefore the series (5) satisfies the hypothesis of the lemma and the conclusion follows.

PROOF OF THE THEOREM. What follows is nothing but a translation in the framework of arbitrary groups of Billard's proof of Proposition A of [1]; the Corollary just proved will be used instead of Lemma 3 of [1]. Suppose that with positive probability (4) does not represent a continuous function, then the argument given in [1, Lemma 5] or [5, Theorem 1] shows that there exist  $\epsilon > 0$  and numbers  $m_k, M_k$  such that  $m_k < M_k < m_{k+1}$  and

$$\Pr \left\{ \left\| \sum_{n=m_k}^{M_k} \epsilon_n d_n \operatorname{tr}(A_n D_n(\xi_n x)) \right\|_\infty > \epsilon \right\} > \epsilon.$$

Let  $U_k = \sum_{n=m_k}^{M_k} \epsilon_n d_n \operatorname{tr}(A_n D_n(\xi_n x))$ , then by the reasoning made in [6, p. 220],

$$(6) \quad \sum_{k=1}^\infty U_k \in L^\infty \quad \text{with probability one.}$$

But then, applying the Corollary to the series in (6) we obtain

$$(7) \quad \limsup_p \left\| \sum_{k=1}^p U_k \right\|_\infty < \infty \quad \text{with probability one.}$$

Let  $\Omega$  be the probability space on which  $\epsilon_n$  and  $\xi_n$  are defined. We introduce probability spaces  $\Omega'$  and  $\Omega''$  and random variables  $\epsilon'_k$  and  $\xi''_k$  defined in a similar way on  $\Omega'$  and  $\Omega''$ . Using Fubini's theorem as in [1, p. 313] and the fact that for each  $\omega' \in \Omega'$  and each  $\omega'' \in \Omega''$ ,  $\xi''_k(\omega'')\xi_n(\omega)$  and  $\epsilon'_k(\omega')\epsilon_n(\omega)$  are similar, respectively, to  $\xi_n(\omega)$  and  $\epsilon_n(\omega)$ , we obtain that, almost surely with respect to  $\Omega \times \Omega' \times \Omega''$

$$(8) \quad \sum_{k=1}^{\infty} \epsilon'_k U_k(\xi''_k x) \in L^{\infty}$$

and

$$(9) \quad \limsup_{p \rightarrow \infty} \left\| \sum_{k=1}^p \epsilon'_k U_k(\xi''_k x) \right\|_{\infty} < \infty.$$

The events  $\|U_k(x)\|_{\infty} \geq \epsilon$  are independent, therefore almost surely infinitely many of them are simultaneously verified. This implies that for almost every  $\omega \in \Omega$  we can find a sequence  $k_j$  such that  $\|U_{k_j}\|_{\infty} \geq \epsilon$ . Also for almost every  $\omega \in \Omega$  (8) and (9) hold with probability one with respect to  $\Omega' \times \Omega''$ . We fix now  $\omega \in \Omega$  so that (8) and (9) hold almost surely with respect to  $\Omega' \times \Omega''$  and infinitely many open sets  $I_j = \{x \in G : |U_{k_j}(x)| > \epsilon/2\}$  are nonempty. This  $\omega \in \Omega$  will remain fixed in what follows. Let  $J_j = (\xi''_{k_j})^{-1}I_j$ . The  $\Omega''$ -event  $J_j \neq \emptyset$  has probability one and because of the independence of  $\xi''_{k_j}$ , almost surely we can find a point of  $G$  which belongs to infinitely many  $J_j$ . We choose then  $\omega''$  so that (9) holds almost surely with respect to  $\Omega'$  and a point  $x_0$  belongs to infinitely many  $J_j$ . But if (9) holds almost surely, in particular,

$$\limsup_{p \rightarrow \infty} \left| \sum_{k=1}^p \epsilon'_k U_k(\xi''_k x_0) \right| < \infty$$

almost surely (with respect to  $\Omega'$ ;  $U_k(\xi''_k x_0)$  are now constants since  $\omega$  and  $\omega''$  have been fixed). This implies by [6, (8.3), p. 213 and Remark (d), p. 205] that  $\sum |U_k(\xi''_k x_0)|^2 < \infty$  which contradicts the fact that for infinitely many  $k$ ,  $|U_k(\xi''_k x_0)| > \epsilon/2$ .

REMARK 1. The conclusion of the theorem also holds for random Fourier series of the type  $\sum d_i \operatorname{tr}(X_i D_i(x))$  where  $X_i$  are independent random variables such that  $X_i$  is similar to  $-X_i$  and to  $X_i D_i(x)$  for each  $x \in G$ . A proof can be obtained using Fubini's theorem as in [2, Théorème 6].

REMARK 2. In the commutative case the hypothesis that  $\sum \epsilon_i d_i \operatorname{tr}(A_i D_i(\xi; x))$  represents a bounded function for every choice of  $\xi_i \in G$  and  $\epsilon_i = \pm 1$  is sufficient to imply that the series represents a

function belonging to  $L^2(G) * L^2(G)$ . The example given in [3] shows that this may be false in the noncommutative case.

3. **A counterexample.** As remarked in the Introduction, the conclusion of the theorem holds for random Fourier series, defined on a *commutative* compact group  $G$ , of the type

$$\sum_{n=1}^{\infty} \epsilon_n a_n \gamma_n(x)$$

where  $\gamma_n$  are characters of  $G$  and  $a_n$  fixed complex numbers. This fact is an immediate consequence of Theorem 6 of [5, Chapter II] or of the Corollary to the Theorem 3 of [4].

For noncommutative groups the situation is entirely different: in this section we shall construct an example of a compact group  $G$  and a series

$$(10) \quad \sum_{i=1}^{\infty} \epsilon_i \operatorname{tr}(A_i D_i(x)),$$

which represents for every value of  $\epsilon_i$  a bounded function and yet for no such value it represents a continuous function. We let  $H$  be the dihedral group which we represent faithfully as the group of two by two matrices of the type

$$\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \epsilon_1 \\ \epsilon_2 & 0 \end{pmatrix} \text{ with } \epsilon_1, \epsilon_2 = \pm 1.$$

We let  $H_n, n=1, 2, \dots$ , be copies of  $H$  and  $G = \prod_{n=1}^{\infty} H_n$ . Let  $D_n: G \rightarrow H_n$  be the irreducible representation of degree two obtained by projecting  $G$  onto  $H_n$ . Thus if  $x \in G, x = \{x_n\}, x_n \in H_n$ , then  $D_n(x) = x_n$ . Let  $v_1^{(n)}(x)$  be the entry in the upper left corner of the matrix  $x_n$  and  $v_2^{(n)}(x)$  be the entry in the upper right corner. Define

$$\begin{aligned} f_1(x) &= v_1^{(1)}(x), \\ f_2(x) &= v_2^{(1)}(x) v_1^{(2)}(x), \\ &\dots, \\ f_n(x) &= v_2^{(1)}(x) v_2^{(2)}(x) \dots v_1^{(n)}(x). \end{aligned}$$

Then since for each  $k, v_1^{(k)}(x) v_2^{(k)}(x) \equiv 0, f_n(x) f_m(x) \equiv 0$  for  $n \neq m$ . Moreover each  $f_n$  belongs to the translation invariant subspace of  $L^2$  generated by the character of the representation  $\tilde{D}_n = D_1 \otimes D_2 \otimes \dots \otimes D_n$ . Now the representations  $\tilde{D}_n$  are all irreducible, therefore

the series

$$(11) \quad \sum_{n=1}^{\infty} \epsilon_n f_n$$

is a Fourier series of the type of (10). We also have that

$$\sup_p \left\| \sum_{n=1}^p \epsilon_n f_n \right\|_{\infty} \leq 1,$$

because for each  $x$  only one of the  $f_n$  will be such that  $f_n(x) \neq 0$ . Therefore (11) represents a bounded measurable function for every choice of the  $\epsilon_n$ . On the other hand it is easy to see that (11) cannot represent a continuous function  $f$  for any value of  $\epsilon_n$ . Indeed if  $\epsilon_n$  are fixed we define sequences  $x_N$  and  $y_N$  of element of  $G$  by letting

$$D_n(x_N) = D_n(y_N) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } n < N,$$

$$D_n(x_N) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } D_n(y_N) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ if } n \geq N;$$

we have that  $\lim_N y_N = \lim_N x_N$ , but  $f(x_N) = f_N(x_N) = \epsilon_N$  and  $f(y_N) = f_N(y_N) = -\epsilon_N$ . Therefore  $f$  cannot be continuous.

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