THE α-CLOSURE αX OF A TOPOLOGICAL SPACE X

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Introduction. It is known that every completely regular space S has a real compactification νX, contained in βX, with the following property: every $f \in C(X)$ has an extension $f^\nu \in C(\nu X)$ [**2**, p. **118**]. (A space S is real-compact iff every Z-ultrafilter with the countable intersection property is fixed. A real compactification of X is a real compact space in which S is densely imbedded.)

This paper is a study of α-spaces (every open ultrafilter with the countable intersection property converges). The main results are contained in §III. We show that for any space X, there exists an α-closure (see Definition 3.7) αX of X, contained in κX [**3**, p. 89], with the following property: If Y is any other α-closure of S and $i: X \to Y$ is the inclusion, then i can be extended continuously to a function $\bar{i}: \alpha X \to Y$.

Since the construction of αX is based on κX and the structure of κX is related to open ultrafilter, therefore in §I, we study open ultrafilters and in §II, we state a main theorem about κX.

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I. **Open ultrafilters.** Throughout this paper, X denotes a topological space. For any $A \subset X$, we denote the closure of A in X by $\bar{A}$ or $\mathrm{Cl}_X A$.

1.1 Definition. An *open filter base* is a filter base consisting exclusively of open sets. An *open filter* is a nonempty collection of *open* sets say $\mathfrak{U}$ satisfying the following properties:

(a) $\emptyset \notin \mathfrak{U}$.

(b) If $U_1, U_2 \in \mathfrak{U}$, then $U_1 \cap U_2 \in \mathfrak{U}$.

(c) If $U \in \mathfrak{U}$ and G is open, $G \supset U$, then $G \in \mathfrak{U}$.

An *open ultrafilter* is an open filter which is maximal in the collection of open filters.

We will state but omit the proof of the following lemmas.

1.2 Lemma. *If $\mathfrak{U}$ is an open filter on X, the following hold*:

(1) *$\mathfrak{U}$ is an open ultrafilter on X iff for any open set G such that $G \cap U \neq \emptyset$ for all $U \in \mathfrak{U}$, then $G \in \mathfrak{U}$.*

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(2) *$\mathcal{U}$ is an open ultrafilter on X iff for any open set G such that $G \notin \mathcal{U}$, then $(X-\overline{G}) \in \mathcal{U}$.*

(3) *If $\mathcal{U}$ is an open ultrafilter, then p is a cluster point of $\mathcal{U}$ iff $\mathcal{U} \to p$. ($\mathcal{U}$ converges to p.)*

1.3 LEMMA. *Suppose X is a dense subset of Y and $\mathcal{U}'$ is an open ultrafilter on Y. Let $\mathcal{U} = \mathcal{U}' \cap X = \{U' \cap X : U' \in \mathcal{U}'\}$. Then $\mathcal{U}$ is an open ultrafilter on X. Moreover, $\mathcal{U} \to p$ iff $\mathcal{U}' \to p$.*

1.4 LEMMA. *Suppose X is a dense subset of a topological space Y, and $\mathcal{U}$ is an open ultrafilter on X. Let $\mathcal{U}' = \{G : G$ open in Y and $G \cap X \in \mathcal{U}\}$. Then $\mathcal{U}'$ is an open ultrafilter on Y. Moreover, $\mathcal{U}' \to p$ in Y iff $\mathcal{U} \to p$ in Y.*

1.5 COROLLARY. *If X is an open, dense subset of Y, then $\mathcal{U}$ and $\mathcal{U}'$ as above are related as follows:*

$$\mathcal{U} = \{U \in \mathcal{U}' : U \subset X\}.$$

II. The absolute closure κX of a topological space X.

2.1 DEFINITION. A Hausdorff space X is called absolutely closed if X is closed in every Hausdorff space in which it is imbedded. Or equivalently, every open filter on X has a cluster point ([**1**, p. 160]; or [**3**, p. 88]).

2.2 DEFINITIONS. Let X, Y be Hausdorff spaces such that X is dense in Y and Y is absolutely closed. We call Y an *absolute closure* of X. An absolute closure Y of X is called a *largest absolute closure* of X, if for any other absolute closure say T of X, and $i: X \to T$ is the injection, then there exists $\bar{i}: Y \to T$ such that $\bar{i} | X = i$.

2.3 THEOREM [**3**, p. 89]. *For any Hausdorff space X, there exists a Hausdorff space κX which is a largest absolute closure of X. Moreover, κX is essentially unique.*

III. α-spaces and α-closure.

3.1 DEFINITION. Let $\mathcal{U}$ be a family of subsets of X. We say $\mathcal{U}$ has the *countable intersection property* if for any collection of countable sets $U_n \in \mathcal{U}$, $\bigcap_n U_n \neq \emptyset$. We abbreviate it as c.i.p.

3.2 LEMMA. *The following are equivalent.*

(1) *Every open filter base in X with the countable intersection property has a cluster point.*

(2) *Every open cover of X has a countable dense subsystem.*

3.3 DEFINITION. A topological space X is called an *α-space* if every open ultrafilter with the countable intersection property converges.

3.4 REMARKS. (1) Obviously, every absolutely closed space is an α-space.

(2) Every Lindelöf space satisfies (2) in 3.3, hence it is an α-space. In particular, N is an α-space.

(3) Since 2nd countable implies Lindelöf, therefore every space with a countable base of open sets is an α-space. In particular, R is an α-space.

3.5 LEMMA. *Let T and Y contain X as a dense subset. Further, Y is an α-space. Suppose h is continuous from T into Y whose restriction on X is the inclusion i. Then h extends to a continuous mapping from αT into Y.*

PROOF. Let $\mathcal{P}\in\alpha T-T$, then $\mathcal{P}$ is a nonconvergent open ultrafilter on T with the c.i.p. Let $\mathfrak{U}=\mathcal{P}\cap X$, and $\mathcal{P}'=\{U \text{ open in } Y\colon U'\cap X\in\mathfrak{U}\}$. Suppose $\bigcap_n U_n'=\varnothing$ for some countable collection $U_n'\in\mathcal{P}'$. Since $h^{-1}(U_n')\cap X=U_n'\cap X\in\mathfrak{U}$, by the maximality of $\mathcal{P}$ and 1.4, we have $h^{-1}(U_n')\in\mathcal{P}$. This gives $\bigcap_n h^{-1}(U_n')=h^{-1}(\bigcap_n U_n')=\varnothing$. Thus $\mathcal{P}'$ is an open ultrafilter on Y with the c.i.p. Since Y is an α-space, there exists $p\in Y$ such that $\mathcal{P}'\to p$ in Y. Define $f(\mathcal{P})=p$ for $\mathcal{P}\in\alpha T-T$ and $f(t)=h(t)$ for $t\in T$. It is clear that f is continuous at each $t\in T$ because T is open in αT. Consider $\mathcal{P}\in\alpha T-T$, and let W be an open neighborhood of p in Y where $f(\mathcal{P})=p$ and $\mathcal{P}\to p$ as described above. Then $W\in\mathcal{P}'$ and thus $W\cap X\in\mathfrak{U}$. It follows that $h^{-1}(W)\cap X\in\mathfrak{U}$ and consequently $h^{-1}(W)\in\mathcal{P}$. Write $G=h^{-1}(W)$, then $G\cup\{\mathcal{P}\}$ is an open neighborhood of $\mathcal{P}$ in αT such that $f(G\cup\{\mathcal{P}\})=h(G)\cup\{p\}\subset W$. Thus f is continuous.

3.6 DEFINITION. Let X, Y be topological spaces, we call Y an *α-closure* of X if

(1) X is dense in Y.

(2) Y is an α-space.

3.7 DEFINITION. Let X be dense in T. We say *T has property α relative to X* if for any α-closure Y of X and $i\colon X\to Y$ is the inclusion, then there exists continuous $\bar{i}\colon T\to Y$ such that $\bar{i}|X=i$.

3.8 THEOREM. *Let X be a dense subset of T, and T has property α relative to X, then $T\subset\kappa X$ (up to a homeomorphism).*

PROOF. By definition, κT is an absolute closure of X. Let Y be any absolute closure of X and $i\colon X\to Y$ be the inclusion. By hypothesis, there exists continuous $h\colon T\to Y$ such that $h|X=i$. Lemma 3.5 is still valid if Y is absolutely closed and αT is replaced by κT. Hence we can extend h continuously to a f from κT onto Y. By the uniqueness of κX, we conclude that $\kappa X\approx\kappa T$. Thus $T\subseteq\kappa X$.

We are looking for an α-closure of X with the property α relative to X. By the above theorem, such space must lie between X and κX.

3.9 THEOREM. *Let $\alpha X^\nu = \{\mathcal{P} : \mathcal{P}$ is a nonconvergent open ulatrafilter on X with the c.i.p.$\}$. Define αX*[2] *$= X \cup \alpha X^\nu$ as a subspace of κX. Then the following hold*:

(1) *αX is an α-closure of X.*

(2) *αX has property α relative to X.*

Moreover, αX is essentially unique with respect to the above properties.

PROOF. Clearly X is dense and open in αX. Now let $\mathcal{U}$ be any open ultrafilter in αX with the c.i.p. We will show $\mathcal{U}$ converges in αX. This will complete the proof of (1). Let $\mathcal{P} = \mathcal{U} \cap X = \{U \cap X : U \in \mathcal{U}\}$. Since X is open in αX, by 1.5, $\mathcal{P} \subset \mathcal{U}$ and therefore $\mathcal{P}$ has the c.i.p. If $\mathcal{P} \to x \in X$, then $\mathcal{U} \to x$ in αX by 1.3. If $\mathcal{P}$ is nonconvergent in X, then $\mathcal{P} \in \alpha X^\nu$. We will show $\mathcal{U} \to \mathcal{P}$ in αX.

Let W be an open neighborhood of $\mathcal{P}$ in αX. By the induced topology of κX, we can write $W = G \cup \{\mathcal{P}\}$ where $G \in \mathcal{P}$. Therefore $G \in \mathcal{U}$. This implies W meets every member of $\mathcal{U}$. Thus $\mathcal{P}$ is a cluster point of $\mathcal{U}$. It follows $\mathcal{U} \to \mathcal{P}$ in αX. For (2), let Y be an α-closure of X and $i: X \to Y$ be the inclusion. By 3.5, we can extend i continuously to a map f from αX into Y. Thus αX has property α relative to X. To show αX is essentially unique, suppose T also has properties (1) and (2). Then the identity mapping i on X, which is continuous into T, has a continuous extension f from αX to T. Similarly, it has a continuous extension g from T to αX. It follows that $T \approx \alpha X$ (T is homeomorphic to αX). (See [**2**, p. 5], also [**3**, Lemma 1.16].)

3.10 THEOREM. *The following hold*: (1) *$\alpha X = X$ iff X is an α-space.*

(2) *αX is the largest subspace of κX containing X as a dense subset and having property α relative to X.*

(3) *αX is the smallest α-space between X and κX.*

PROOF. (1) If X is an α-space, then $\alpha X^\nu = \varnothing$. This implies $\alpha X = X$.

(2) Let T be a subspace of κX such that T contains X as a dense subset and T has property α relative to X. Again, it follows from 3.5 that αT has property α relative to X. By the uniqueness of αX, we conclude that $\alpha X \approx \alpha T$. Thus $T \subseteq \alpha X$.

(3) If $X \subset T \subset \kappa X$ and T is an α-space, by the uniqueness of κX, it is easy to see that $\kappa X \approx \kappa T$. By (2), $T = \alpha T$ is the *only* subspace of κT containing T which has property α relative to T. Obviously

[2] If X is not Hausdorff; then by the induced topology of κX, it is clear that αX is Hausdorff except for X [**3**, p. 91].

$T \cup \alpha X$ ($\subset \kappa T$) contains T as an open dense subset and has property α relative to T. Therefore $T = T \cup \alpha X$, which implies $\alpha X \subset T$.

3.11 THEOREM. *Let X be a dense subset of T. The following are equivalent*:

(1) *T has property α relative to X.*
(2) $X \subset T \subset \alpha X$.
(3) $\alpha X \approx \alpha T$.

PROOF. (1) implies (3). If T has property α relative to X, then αT has property α relative to X. Thus by the uniqueness of αX, we conclude that $\alpha X \approx \alpha T$.

(3) implies (2). Obvious.

(2) implies (1). If $i: X \to Y$ is the inclusion where Y is any α-closure of X, then there exists a continuous mapping $f: \alpha X \to Y$ such that $f|X = i$. Now $g = f|T$ will be the desired extension of i on T.

3.12 REMARK. Here is a question that I am not able to answer yet: Does there exist a topological space which is not an α-space? Note that for discrete spaces, the above question is equivalent to that of the existence of measurable cardinals.

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