COMPACTIFICATIONS OF HAUSDORFF SPACES

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1. **Introduction.** In this paper, we describe methods of imbedding a Hausdorff space X in a compact space $\overline{X}$ so that each function in a given family of continuous functions on X has a continuous extension to $\overline{X}$ and the family of extensions separates the points of $\overline{X}-X$. In particular, if X is completely regular but not locally compact, then we shall exhibit a non-Hausdorff compactification which contains X as an open subset and is bigger than the Stone-Čech compactification of X. (Of course, every compactification of X is non-Hausdorff if X is not completely regular.) We shall also show that the completion of a metric space M may be obtained as a subset of a compactification of M by a rather simple construction.

By a compactification of a Hausdorff space X, we mean a compact space $\overline{X}$ which contains, as a dense subset, the image of X under a fixed homeomorphism f. We usually do not distinguish between X and $f(X)$, and we say that $\overline{X}$ contains X as a dense subset. In what follows, X is always a noncompact Hausdorff space, $\Delta\overline{X}$ denotes the closure of $\overline{X}-X$ in $\overline{X}$, and a mapping is always a continuous function. If $\overline{X}$ is Hausdorff ,we say that $\overline{X}$ is a Hausdorff compactification of X. If $\overline{X}$ is not Hausdorff, however, we still assume that it satisfies the following properties:

I. Compact subsets of X are closed in $\overline{X}$.

II. Any two distinct points x and y in $\Delta\overline{X}$ can be separated by disjoint open sets; i.e., there exist open sets U and V in $\overline{X}$ with $x\in U$, $y\in V$, and $U\cap V=\emptyset$.

III. For each point $x\in X$ there is at most one point $z\in\overline{X}-X$ such that x and z cannot be separated by disjoint open sets in $\overline{X}$.

Clearly, II and III are necessary conditions for the points in $\Delta\overline{X}$ to be separated by a family of continuous functions from $\overline{X}$ into a Hausdorff space; we shall show later that, together with Condition I, they are also sufficient. The following properties of $\overline{X}$ are consequences of I, II, and the fact that X is dense in $\overline{X}$.

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PROPOSITION 1.1. *If* $\overline{X}$ *is a compactification of* X, *then*:

(i) *Two distinct points in* X *can be separated by disjoint open sets in* $\overline{X}$.

(ii) *The space* $\overline{X}$ *is* T_1.

(iii) *In* $\overline{X}$, *a sequence converges to at most one point.*

(iv) *If* X *satisfies the first axiom of countability then compact subsets of* $\overline{X}$ *are closed.*

(v) *If the compactification* $\overline{X}$ *satisfies the first axiom of countability, then* $\overline{X}$ *is Hausdorff.*

(vi) X *is locally compact if and only if* X *is open in* $\overline{X}$ *and* $\overline{X}$ *is Hausdorff.*

EXAMPLE. Following Arens [1], let X be the set of all pairs of nonnegative integers such that each point other than (0, 0) is an open set and every neighborhood of (0, 0) contains all but a finite number of points in all but a finite number of columns C_n, where $C_n = \{(n, m): m \in \boldsymbol{Z}^+\}$. Then X is not first countable. Let $\overline{X}$ be the space X together with a point P whose neighborhoods omit at most a finite number of points in X. Any nonrepeating sequence with an infinite number of points in each column C_n converges to P and has cluster point (0, 0). Moreover, $\overline{X} - \{(0, 0)\}$ is compact but not closed in $\overline{X}$.

Now let $\overline{X}$ and $\tilde{X}$ be two compactifications of X. By the notation $\overline{X} \geqq \tilde{X}$ we mean there is a mapping T of $\overline{X}$ onto $\tilde{X}$ such that $T|X$ is the identity map. (To be more accurate, we should say that if f and g are the homeomorphisms of X into $\overline{X}$ and $\tilde{X}$ respectively, then $T \circ f = g$.) If we also have $\tilde{X} \geqq \overline{X}$, then T is a homeomorphism, and in this case we write $\overline{X} \cong \tilde{X}$.

2. Q-**compactifications of** X. Let Q be a nonvoid family of continuous functions on X with each $f \in Q$ having its range contained in a compact Hausdorff space S_f. Using the methods of [5], we now describe a compactification of X which is the compactification defined in [5] when X is locally compact.

DEFINITION. Let Y be the product space $\prod_{f \in Q} S_f$ and e the evaluation map sending X into Y. (For each $x \in X$, $e(x)(f) = f(x)$.) Set

$$\Delta = \bigcap \{\overline{e(X - K)}: K \text{ compact}, K \subset X\},$$

and let $\overline{X}^Q$ be the (disjoint) union $X \cup \Delta$. Given an open set U in Y and a compact set $K \subset X$, we set $U_K = [U \cap \Delta] \cup [e^{-1}(U) - K]$. If $\mathfrak{J}$ is the topology on $\overline{X}^Q$ generated by the base consisting of all open sets in X and all the sets U_K, then $(\overline{X}^Q, \mathfrak{I})$ is called the Q-compactification of X.[2]

It is not hard to show that $\overline{X}^Q$ is, indeed, a compactification of X; e.g., $\overline{X}^Q$ is compact since a net which is eventually in the complement of every compact subset of X has a cluster point in Δ. Clearly, $\overline{X}^Q$ also has the following properties:

THEOREM 2.1. *Each function $f\in Q$ has a continuous extension mapping $\overline{X}^Q$ into S_f, and the family of these extensions separates the points in $\overline{X}^Q-X$. Moreover, X is open in $\overline{X}^Q$. Thus, if X is not locally compact, $\overline{X}^Q$ is neither Hausdorff nor a space which satisfies the first axiom of countability.*

Next we show that these properties determine the compactification $\overline{X}^Q$ up to a homeomorphism.

THEOREM 2.2. *Let $\tilde{X}$ be a compactification of X such that each function f in a nonvoid subfamily Q_0 of Q has a continuous extension mapping $\tilde{X}$ into S_f and these extensions separate the points in $\Delta\tilde{X}$. Then $\overline{X}^Q \geqq \tilde{X}$. If, moreover, X is open in $\tilde{X}$ (e.g., if X is locally compact) and if $Q_0=Q$, then $\overline{X}^Q \cong \tilde{X}$.*

PROOF. Let $\Gamma=\Delta\tilde{X}$ and recall that $\Delta=\overline{X}^Q-X$. Let e, $\bar{e}$, and $\tilde{e}$ be the evaluation maps sending X, $\overline{X}^Q$ and $\tilde{X}$ respectively into the product space $Y_0=\prod_{f\in Q_0}S_f$. Given $x_0\in\Delta$, let N be a neighborhood of $\bar{e}(x_0)$ in Y_0, and let $\overline{N}$ be its closure in Y_0. Then $\tilde{e}^{-1}(\overline{N})\cap\Gamma\neq\varnothing$, for otherwise $e^{-1}(\overline{N})=\tilde{e}^{-1}(\overline{N})$ is a compact subset of X, and $\bar{e}^{-1}(N)-e^{-1}(\overline{N})$ is a neighborhood of x_0 in $\overline{X}^Q$ that does not intersect X. Let $T(x_0)$ be the unique point in the intersection of all sets of the form $\tilde{e}^{-1}(\overline{N})\cap\Gamma$ where N ranges over the neighborhood system of $\bar{e}(x_0)$ in Y_0. One thus extends the identity mapping on X to a function T from $\overline{X}^Q$ into $\tilde{X}$, and in a similar way one shows that T is onto. Clearly, T is continuous at each point of X. Given $x_0\in\Delta$ and U an open neighborhood of $T(x_0)$ in $\tilde{X}$, there is an open set V in Y_0 such

[2] (Added July 19, 1968.) The author has learned of a manuscript, *Minimum and maximum compactifications of arbitrary topological spaces*, by R. F. Dickman, Jr., submitted in January 1967 to the Trans. Amer. Math. Soc. Using a different definition than the one given here and starting with an arbitrary topological space X and a collection of mappings from X into a single compact Hausdorff space S, Professor Dickman has proved the existence of a compactification $\alpha_Q X$ which has the properties established for $\overline{X}^Q$ by Theorems 2.1 and 2.2 below, and is thus equivalent to $\overline{X}^Q$. Professor Dickman has informed the author that these results will be included in a revised paper entitled *Compactifications and real-compactifications of arbitrary topological spaces*.

that $\tilde{e}[\Gamma - U] \subset V$ and $\tilde{e}(T(x_0)) = \tilde{e}(x_0) \notin \overline{V}$. Moreover, the set $K = \tilde{X} - [U \cup \tilde{e}^{-1}(V)]$ is a compact subset of X. Let $W = \overline{X}^Q - [K \cup \tilde{e}^{-1}(\overline{V})]$. Then $T(W) \subset U$, so T is continuous at x_0, and thus T is continuous on all of $\overline{X}^Q$. The rest of the proof is clear.∎

COROLLARY 2.3. *If Q_0 is a nonvoid subset of Q, then $\overline{X}^Q \geqq \overline{X}^{Q_0}$.*

Let $\bar{e}$ be the evaluation map sending $\overline{X}^Q$ into Y; then

$$\bar{e}(\overline{X}^Q) = \overline{e(X)},$$

since $\bar{e}(\overline{X}^Q) = e(X) \cup \Delta$ in Y. One can, moreover, easily establish the following result:

PROPOSITION 2.4. *If there are no compact neighborhoods in X, then Δ is the closure of $e(X)$.*

EXAMPLES. (1) If Q consists of one mapping to a one point space, then $\overline{X}^Q$ is the Alexandroff one point compactification of X. (See [4, p. 150].)

(2) Let X be the rational numbers in the real unit interval $[0, 1]$, and let Q consist of the single function $f(x) = x$. Then Δ is homeomorphic to $[0, 1]$. A typical neighborhood of a point $y_0 \in \Delta$ is given by a constant $\epsilon > 0$ and a compact subset K of X; it has the form

$$\{y \in \Delta: |y - y_0| < \epsilon\} \cup \{x \in X - K: |x - y_0| < \epsilon\}.$$

If $\tilde{X} = [0, 1]$, then $\overline{X}^Q \geqq \tilde{X}$, but we do not have $\overline{X}^Q \leqq \tilde{X}$.

(3) Let H be an infinite-dimensional Hilbert space with the norm topology and let H^* be its dual space with the weak topology. Let X be the closed unit ball in H, Q be the functions in H^*, and e be the canonical map sending H onto H^*. By Proposition 2.4, $\Delta = e(X)$, which is the closed unit ball with the weak topology. A typical neighborhood of a point $x \in \Delta$ has the form $[N \cap \Delta] \cup [e^{-1}(N) \cap X - K]$, where N is a weak neighborhood of x in H^* and $K \subset X$ is compact in the norm topology.

Finally, we note that the results of this section can be applied to an arbitrary topological space X if one works with closed and compact subsets of X instead of compact subsets of X. The details are left to the reader.

3. **Hausdorff Q-compactifications.** In this section, we assume that X is homeomorphic to its image $e(X)$ in the product space $Y = \prod_{f \in Q} S_f$ (see [4, p. 116]); Q is the family of functions described in §2. (If X is locally compact, then, following Constantinescu and Cornea [3], one may adjoin all continuous real-valued functions

with compact support to a given family of continuous functions to obtain a Q which satisfies these assumptions.) Identify X with $e(X)$; as is well known [7], [2], [4] the closure of $e(X)$ in Y is a compact Hausdorff space which contains X (i.e., $e(X)$) as a dense subset, and the functions in Q have continuous extensions to the closure of $e(X)$. All the points in the closure of $e(X)$ are separated by these extensions. We call this closure the Hausdorff Q-compactification of X, and we denote it by $\overline{X}^{HQ}$. By Theorems 2.1 and 2.2, $\overline{X}^{Q} \geqq \overline{X}^{HQ}$, and $\overline{X}^{Q} \cong \overline{X}^{HQ}$ if and only if X is locally compact. On the other hand, if there are no compact neighborhoods in X, then by Proposition 2.4, $\Delta = \Delta\overline{X}^{Q}$ is homeomorphic to $\overline{X}^{HQ}$. The space $\overline{X}^{HQ}$ is unique in the following sense:

THEOREM 3.1. *Let X be a Hausdorff compactification of X with each function in Q having a continuous extension mapping $\tilde{X}$ into S_f. If these extensions separate the points of $\tilde{X}-X$, then $\overline{X}^{HQ} \cong \tilde{X}$.*

PROOF. We need only show that the evaluation map $\tilde{e}$ which sends $\tilde{X}$ onto $\overline{X}^{HQ}$ is injective. Assume that $\tilde{e}(x)=\tilde{e}(y)$ for some $x \in X$ and $y \in \tilde{X}-X$. Let $U \subset \tilde{X}$ be a neighborhood of y such that $x \notin \overline{U}$, and let $C=\overline{U} \cap X$. Then C is closed in X, so $\tilde{e}(C)=X \cap D$ where D is closed in $\overline{X}^{HQ}$. Since $\tilde{e}(x)$ is not in D, y is not in the closed set $\tilde{e}^{-1}(D)$. But this is impossible since y is in the closure of $C=\tilde{e}^{-1}(D) \cap X$. Thus, $\tilde{e}$ is injective and therefore a homeomorphism.∎

Note that if Q_0 is a nonvoid subset of Q and X is homeomorphic to its image in $\prod_{f \in Q_0} S_f$, then since the projection of $\prod_{f \in Q} S_f$ onto $\prod_{f \in Q_0} S_f$ is continuous, $\overline{X}^{HQ} \geqq \overline{X}^{HQ_0}$.

EXAMPLES. (1) If X is the set of rational numbers in $[0, 1]$ and Q consists of the single function $f(x)=x$, then $\overline{X}^{HQ}=[0, 1]$. (Compare with Example 2 of §2.)

(2) Let X be a metric space with metric d, and let Q be the family of functions $\{d_x \colon x \in X$, where $d_x(y)=d(x, y)$ for all $y \in X\}$. Each d_x has its range in the interval $[0, +\infty]$. Set $X^*=\{z \in \overline{X}^{HQ} \colon \forall \epsilon > 0$ $\exists x \in X$ with $d_x(z) < \epsilon\}$, and let $d^*(z, w)=\inf_{x \in X}[d_x(z)+d_x(w)]$ for each pair (z, w) in $X^* \times X^*$. Then one can show that d^* is a metric which generates the relative product topology on X^* and (X^*, d^*) is the completion of (X, d).

(3) A similar construction gives the completion X^* of a Hausdorff uniform space X: If the uniform topology of X is generated by the family of pseudometrics $\{d_\alpha \colon \alpha \in A\}$, and $Q=\{d_\alpha(x, \cdot) \colon \alpha \in A, x \in X\}$, then

$$X^* = \{z \in \overline{X}^{HQ} \colon \forall \epsilon > 0 \text{ and } \forall \alpha \in A, \exists x \in X \text{ with } d_\alpha(x, z) < \epsilon\}.$$

Using filters, Samuel [6] has constructed the largest compactification X^v in which a given uniform space X can be uniformly imbedded; the completion X^* is the subset of X^v consisting of all limits of Cauchy ultrafilters in X. However if Q is any collection of uniformly continuous functions from X into the real unit interval I such that $\overline{X}^{HQ}$ exists, then X is uniformly imbedded in $\overline{X}^{HQ}$. Moreover, any Hausdorff compactification $\tilde{X}$ in which X is uniformly imbedded is of the form $\overline{X}^{HQ}$ where each $f\in Q$ maps X uniformly into I. (See Theorem 4.2.) It follows that $\overline{X}^v\cong\overline{X}^{H\mathfrak{U}}$ where $\mathfrak{U}$ is the set of all uniformly continuous mappings of X into I. Thus the compactifications used in the last two examples are, in general, smaller than X^v. If, for example, X is the real line with 0 removed and X has the additive uniform structure, then the compactification used in Example 2 is the one point compactification of the *real line* where as X^v is "a space almost as complicated as the Čech compactification of the real line" [6, p. 124].

4. **Properties of arbitrary compactifications.** Let $\tilde{X}$ be any compactification of the Hausdorff space X such that $\tilde{X}$ satisfies the three conditions in §1. Let $R\subset\tilde{X}\times\tilde{X}$ be the equivalence relation which consists of the diagonal set $\{(x, x): x\in\tilde{X}\}$ together with all pairs $(x, y)\in\tilde{X}\times\tilde{X}$ for which there is a $z\in\tilde{X}-X$ such that neither x nor y can be separated from z by disjoint open sets. As usual, $R[x]$ denotes the set of all points in $\tilde{X}$ equivalent to a point x, and for any set $A\subset\tilde{X}$, $R[A]=\bigcup_{x\in A}R[x]$.

PROPOSITION 4.1. *The relation R has the following properties*:

(i) *For each $x\in\tilde{X}$, $R[x]$ is closed and therefore compact.*

(ii) *If x and y are points in $\tilde{X}$ with $R[x]\cap R[y]=\varnothing$, then there are disjoint open sets U and V in $\tilde{X}$ with $R[x]\subset U$ and $R[y]\subset V$.*

(iii) *If $z\in\tilde{X}-X$ and U is an open neighborhood of z, then $R[z]$ is contained in the closure $\overline{U}$ of U.*

(iv) *If $x\in X\cap\Delta\tilde{X}$, then $R[x]=\{x\}$.*

(v) *If C is compact in $\tilde{X}$, then $R[C]$ is closed.*

PROOF. We shall only prove (v). We show first that $R[C]\cap\Delta\tilde{X}$ is closed. If $\{z_\alpha\}_{\alpha\in A}$ is a net in $R[C]\cap\Delta\tilde{X}$ and $\{z_\alpha\}$ converges to $z\in\Delta\tilde{X}$, then for each α in the index set A there is a point x_α in $R[z_\alpha]\cap C$. Let $x\in C$ be a cluster point of the net $\{x_\alpha\}_{\alpha\in A}$. Given open neighborhoods U and V of z and x respectively, there is an $\alpha\in A$ such that $z_\alpha\in U$ and $x_\alpha\in V$. Since $x_\alpha\in R[z_\alpha]$ and either $x_\alpha=z_\alpha$ or $z_\alpha\in\tilde{X}-X$, it follows that $U\cap V\neq\varnothing$, and thus $z\in R[C]$. We have shown that $R[C]\cap\Delta\tilde{X}$ is closed.

Assume now that $y_0 \notin R[C]$. Then for each set $R[x] \subset R[C]$, there is a pair of disjoint open sets U and V in $\tilde{X}$ with $R[x] \subset U$ and $y_0 \in V$. Thus the compact set $C \cup [R[C] \cap \Delta \tilde{X}]$ is contained in a finite union of open sets $\{U_i : i = 1, 2, \cdots, n\}$ such that $y_0 \notin \bigcup_{i=1}^n \overline{U}_i$. But by (iii), $R[C]$ is contained in $\bigcup_{i=1}^n \overline{U}_i$. Thus $R[C]$ is closed.■

We next show that $\tilde{X}/R$ is Hausdorff; clearly, R is the finest relation for which this can be true. It follows that the arbitrarily chosen compactification $\tilde{X}$ is comparable with an appropriate Q-compactification. The following theorem for the case that $\tilde{X}$ is Hausdorff is due to Čech [2].

THEOREM 4.2. *Let $\tilde{Q}$ be the set of all mappings of $\tilde{X}$ into the unit interval $[0, 1]$, and let Q be the set of restrictions $\{f \mid X : f \in \tilde{Q}\}$. Then $\tilde{Q}$ separates the points in $\Delta \tilde{X}$, and thus $\overline{X}^Q \geqq \tilde{X}$. If X is open in $\tilde{X}$, then $\overline{X}^Q \cong \tilde{X}$. If $\tilde{X}$ is Hausdorff, then $\overline{X}^{HQ} \cong \tilde{X}$.*

PROOF. By Theorems 2.2 and 3.1, we need only show that $\tilde{Q}$ separates the points in $\Delta \tilde{X}$. Let P be the projection of $\tilde{X}$ onto the quotient space $\tilde{X}/R$. If $P(x)$ and $P(y)$ are distinct points in $\tilde{X}/R$, then there are disjoint neighborhoods U and V of $R[x]$ and $R[y]$ in $\tilde{X}$. Let $C = X - U$ and $D = X - V$. Then $R[C]$ is a closed set with $R[C] \cap R[x] = \varnothing$; $R[D]$ is a closed set with $R[D] \cap R[y] = \varnothing$, and $R[D] \cup R[C] = \tilde{X}$. Therefore, $P(R[C])$ and $P(R[D])$ are closed sets in $\tilde{X}/R$ with $P(x) \notin P(R[C])$, $P(y) \notin P(R[D])$, and $P(R[C]) \cup P(R[D]) = \tilde{X}/R$. Thus $\tilde{X}/R$ is Hausdorff, and the theorem follows from Urysohn's lemma.■

COROLLARY 4.3. *Every compactification of a locally compact Hausdorff space is a Q-compactification. Every Hausdorff compactification of a completely regular space is a Hausdorff Q-compactification.*

Finally, we let $\mathfrak{I}$ be the set of all mappings of X into the unit interval $[0, 1]$, and we consider the compactifications $\overline{X}^{\mathfrak{I}}$ and $\overline{X}^{H\mathfrak{I}}$. Of course, $\overline{X}^{H\mathfrak{I}}$ is only defined if X is completely regular, and it is the Stone-Čech compactification of X.

THEOREM 4.4. *Let $\tilde{X}$ be any compactification of X. Then $\overline{X}^{\mathfrak{I}} \geqq \tilde{X}$, and as is well known, $\overline{X}^{H\mathfrak{I}} \geqq \tilde{X}$ if $\tilde{X}$ is Hausdorff.*

PROOF. The result follows from Theorem 4.2, Corollary 2.3 and the remark following Theorem 3.1.■

We have shown that if X is completely regular, then $\overline{X}^{\mathfrak{I}} \geqq \overline{X}^{H\mathfrak{I}}$, while $\overline{X}^{\mathfrak{I}} \cong \overline{X}^{H\mathfrak{I}}$ only if X is locally compact. If X is not locally compact, then $\overline{X}^{\mathfrak{I}}$ dominates a larger class of compactifications than the Stone-Čech compactification $\overline{X}^{H\mathfrak{I}}$. Indeed if X is not completely regular, $\overline{X}^{Hs}$ is not even defined. Moreover, X is always an open subset of $\overline{X}^{s}$, but only when X is locally compact is it open in $\overline{X}^{Hs}$.

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