

A NOTE ON TOPOLOGICAL PARALLELIZABILITY

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A differentiable manifold M is said to be parallelizable if the tangent vector bundle of M is trivial. A topological manifold M is said to be topologically parallelizable if the tangent microbundle of M is trivial. In [2] Milnor has shown that on some open set M in some Euclidean space \mathbf{R}^n there exists a differentiable structure with respect to which the integral Pontrjagin class $p(M)$ of M is different from 1. It follows that on a topologically parallelizable manifold it is possible to have a differentiable structure with respect to which the manifold is not parallelizable.

It is known that the only spheres (of dimension ≥ 1) which are differentially parallelizable are S^1 , S^3 and S^7 [1]. It is also known that the only spheres which have fibre homotopically trivial tangent sphere bundles are S^1 , S^3 and S^7 [3].

In this note we prove

PROPOSITION 1. For every integer $q \geq 1$ the map

$$\prod_{q-1} (O(q)) \xrightarrow{i_*} \prod_{q-1} (\text{Top}(q))$$

is a monomorphism.

Here $\text{Top}(q)$ denotes the group of homeomorphisms of \mathbf{R}^q fixing the origin and $i: O(q) \rightarrow \text{Top}(q)$ the inclusion.

As immediate corollaries we get

COROLLARY 1. Any vector bundle of rank q over the sphere S^q is trivial as a microbundle if and only if it is trivial as a vector bundle.

COROLLARY 2. The only spheres of dimension ≥ 1 which are topologically parallelizable are S^1 , S^3 and S^7 .

It is also known that the only real projective spaces of dimension ≥ 1 which are differentially parallelizable are P^1 , P^3 and P^7 . The following lemma is not difficult to prove.

LEMMA 1. If $\tilde{M} \xrightarrow{p} M$ is a covering manifold of a topological manifold then the pull-back $p^*({}^tM)$ of the tangent microbundle of M is isomorphic to the tangent microbundle of \tilde{M} .

From Lemma 1 and Corollary 2 we immediately get

Received by the editors January 16, 1969.

COROLLARY 3. *The only real projective spaces of dimension ≥ 1 which are topologically parallelizable are P^1 , P^3 and P^7 .*

PROOF OF PROPOSITION 1. We use the following result (not yet published) of Novikov and Siebenmann.

THEOREM (NOVIKOV-SIEBENMANN). *Any vector bundle over S^q is stably trivial as a microbundle if and only if it is stably trivial as a vector bundle.*

This could alternatively be also stated as below: Any vector bundle of rank $\geq q+1$ over the sphere S^q is stably trivial as a microbundle if and only if it is actually trivial as a vector bundle itself.

Let O denote the infinite orthogonal group and Top denote the direct limit of the spaces $\text{Top}(q) \rightarrow \text{Top}(q+1) \rightarrow \dots$ under the natural inclusions. Siebenmann's result above asserts that $\Pi_j(O) \xrightarrow{i_*} \Pi_j(\text{Top})$ is a monomorphism where $i: O \rightarrow \text{Top}$ is the natural inclusion. In the exact sequence

$$(1) \quad \dots \rightarrow \Pi_q(S^q) \xrightarrow{\partial} \Pi_{q-1}(O(q)) \rightarrow \Pi_{q-1}(O(q+1)) \rightarrow \Pi_{q-1}(S^q) = 0$$

corresponding to the fibration $O(q) \rightarrow O(q+1) \rightarrow S^q$ it is known that the image of ∂ is the subgroup L_q of $\Pi_{q-1}(O(q))$ generated by the tangent vector bundle of S^q [4] since $\Pi_{q-1}(O(q+1)) \cong \Pi_{q-1}(O)$ the exact sequence (1) above gives rise to the following exact sequence

$$(2) \quad 0 \rightarrow L_q \rightarrow \Pi_{q-1}(O(q)) \xrightarrow{s_*} \Pi_{q-1}(O) \rightarrow 0$$

where $s: O(q) \rightarrow O$ is the natural inclusion. Denoting the inclusion of $\text{Top}(q)$ in Top by s' we have the following commutative diagram.

$$\begin{array}{ccc} \Pi_{q-1}(O(q)) & \xrightarrow{s_*} & \Pi_{q-1}(O) \\ \downarrow i_* & & \downarrow \iota_* \\ \Pi_{q-1}(\text{Top}(q)) & \xrightarrow{s'_*} & \Pi_{q-1}(\text{Top}) \end{array}$$

The exactness of (2) gives $\text{Ker } s_* = L_q$, and, since $\iota_*: \Pi_{q-1}(O) \rightarrow \Pi_{q-1}(\text{Top})$ is a monomorphism, to prove that $i_*: \Pi_{q-1}(O(q)) \rightarrow \Pi_{q-1}(\text{Top}(q))$ is a monomorphism we have only to prove

LEMMA 2. $i_*|_{L_q}: L_q \rightarrow \Pi_{q-1}(\text{Top}(q))$ is a monomorphism.

Lemma 2 follows from the known facts that the Hopf-Whitehead

J -homomorphism $J: \Pi_{q-1}(O(q)) \rightarrow \Pi_{2q-1}(S^q)$ maps L_q monomorphically into $\Pi_{2q-1}(S^q)$ and that J can be expressed as the composition of

$$\begin{aligned} \Pi_{q-1}(O(q)) &\xrightarrow{i_*} \Pi_{q-1}(\text{Top}(q)) \rightarrow \Pi_{q-1}(H_q) \xrightarrow{\cong} \Pi_{q-1}(A_{q-1}) \\ &\rightarrow \Pi_{q-1}(B_q) \cong {}_2\Pi_{q-1}(S^q), \end{aligned}$$

where H_q and A_{q-1} denote respectively the spaces of homotopy equivalences of $\mathbf{R}^q - o$ and S^{q-1} and B_q denotes the space of homotopy equivalences of the pair (S^q, x_0) .

REMARK. Corollary 2 of this note is an immediate consequence of the result of Milnor-Spanier [3] mentioned earlier. The author is thankful to Professor Browder for bringing this to his notice, and also for the proof of Lemma 2, which replaces a different and slightly longer proof of the author.

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