A NOTE ON TOPOLOGICAL PARALLELIZABILITY

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A differentiable manifold M is said to be parallelizable if the tangent vector bundle of M is trivial. A topological manifold M is said to be topologically parallelizable if the tangent microbundle of M is trivial. In [2] Milnor has shown that on some open set M in some Euclidean space $\boldsymbol{R}^n$ there exists a differentiable structure with respect to which the integral Pontrjagin class $p(M)$ of M is different from 1. It follows that on a topologically parallelizable manifold it is possible to have a differentiable structure with respect to which the manifold is not parallelizable.

It is known that the only spheres (of dimension $\geqq 1$) which are differentiably parallelizable are S^1, S^3 and S^7 [1]. It is also known that the only spheres which have fibre homotopically trivial tangent sphere bundles are S^1, S^3 and S^7 [3].

In this note we prove

PROPOSITION 1. *For every integer $q \geqq 1$ the map*

$$\Pi_{q-1}(O(q)) \xrightarrow{i_*} \Pi_{q-1}(\mathrm{Top}(q))$$

is a monomorphism.

Here Top (q) denotes the group of homeomorphisms of $\boldsymbol{R}^q$ fixing the origin and $i: O(q) \to \mathrm{Top}\ (q)$ the inclusion.

As immediate corollaries we get

COROLLARY 1. *Any vector bundle of rank q over the sphere S^q is trivial as a microbundle if and only if it is trivial as a vector bundle.*

COROLLARY 2. *The only spheres of dimension $\geqq 1$ which are topologically parallelizable are S^1, S^3 and S^7.*

It is also known that the only real projective spaces of dimension $\geqq 1$ which are differentiably parallelizable are P^1, P^3 and P^7. The following lemma is not difficult to prove.

LEMMA 1. *If $\tilde{M} \xrightarrow{p} M$ is a covering manifold of a topological manifold then the pull-back $p^*({}^tM)$ of the tangent microbundle of M is isomorphic to the tangent microbundle of $\tilde{M}$.*

From Lemma 1 and Corollary 2 we immediately get

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Corollary 3. *The only real projective spaces of dimension* $\geqq 1$ *which are topologically parallelizable are* P^1, P^3 *and* P^7.

Proof of Proposition 1. We use the following result (not yet published) of Novikov and Siebenmann.

Theorem (Novikov-Siebenmann). *Any vector bundle over* S^q *is stably trivial as a microbundle if and only if it is stably trivial as a vector bundle.*

This could alternatively be also stated as below: Any vector bundle of rank $\geqq q+1$ over the sphere S^q is stably trivial as a microbundle if and only if it is actually trivial as a vector bundle itself.

Let O denote the infinite orthogonal group and Top denote the direct limit of the spaces Top $(q)\to$Top $(q+1)\to \cdots$ under the natural inclusions. Siebenmann's result above asserts that $\Pi_j(O) \xrightarrow{\iota_*} \Pi_j(\text{Top})$ is a monomorphism where $\iota\colon O\to\text{Top}$ is the natural inclusion. In the exact sequence

$$(1)\qquad \cdots \to \Pi_q(S^q) \xrightarrow{\partial} \Pi_{q-1}(O(q)) \to \Pi_{q-1}(O(q+1)) \to \Pi_{q-1}(S^q) = 0$$

corresponding to the fibration $O(q)\to O(q+1)\to S^q$ it is known that the image of ∂ is the subgroup L_q of $\Pi_{q-1}(O(q))$ generated by the tangent vector bundle of S^q [4] since $\Pi_{q-1}(O(q+1)) \cong \Pi_{q-1}(O)$ the exact sequence (1) above gives rise to the following exact sequence

$$(2)\qquad 0 \to L_q \to \Pi_{q-1}(O(q)) \xrightarrow{s_*} \Pi_{q-1}(O) \to 0$$

where $s\colon O(q)\to O$ is the natural inclusion. Denoting the inclusion of Top(q) in Top by s' we have the following commutative diagram.

$$\begin{array}{ccc} \Pi_{q-1}(O(q)) & \xrightarrow{s_*} & \Pi_{q-1}(O) \\ \downarrow i_* & & \downarrow \iota_* \\ \Pi_{q-1}(\text{Top}(q)) & \xrightarrow{s'_*} & \Pi_{q-1}(\text{Top}) \end{array}$$

The exactness of (2) gives Ker $s_*=L_q$, and, since $\iota_*\colon \Pi_{q-1}(O) \to \Pi_{q-1}(\text{Top})$ is a monomorphism, to prove that $i_*\colon \Pi_{q-1}(O(q)) \to \Pi_{q-1}(\text{Top}(q))$ is a monomorphism we have only to prove

Lemma 2. $i_*|L_q\colon L_q\to \Pi_{q-1}$ (Top(q)) *is a monomorphism.*

Lemma 2 follows from the known facts that the Hopf-Whitehead

J-homomorphism J: $\Pi_{q-1}(O(q)) \to \Pi_{2q-1}(S^q)$ maps L_q monomorphically into $\Pi_{2q-1}(S^q)$ and that J can be expressed as the composition of

$$\Pi_{q-1}(O(q)) \xrightarrow{i_*} \Pi_{q-1}(\mathrm{Top}(q)) \to \Pi_{q-1}(H_q) \xrightarrow{\cong} \Pi_{q-1}(A_{q-1})$$

$$\to \Pi_{q-1}(B_q) \cong \Pi_{2q-1}(S^q),$$

where H_q and A_{q-1} denote respectively the spaces of homotopy equivalences of $\boldsymbol{R}^q - o$ and S^{q-1} and B_q denotes the space of homotopy equivalences of the pair (S^q, x_0).

REMARK. Corollary 2 of this note is an immediate consequence of the result of Milnor-Spanier [**3**] mentioned earlier. The author is thankful to Professor Browder for bringing this to his notice, and also for the proof of Lemma 2, which replaces a different and slightly longer proof of the author.

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