THE STRUCTURE OF O'/Ω OVER LOCAL FIELDS OF CHARACTERISTIC 2

EDWARD A. CONNORS

In this paper V will be a nondefective quadratic space over a field of characteristic 2, $O(V)$ will be the orthogonal group of V, $O^+(V)$ will be the group of rotations, $\Omega(V)$ will be the commutator subgroup of $O(V)$ and $O'(V)$ will be the spinorial kernel (considered as a subgroup of O^+). We will show that $O'=\Omega$ if F is a local field. This result is known if V is isotropic and may be found in [**3**].

We assume as familiar the theory of quadratic forms over fields of characteristic 2 as treated in [**1**] and [**3**]. In §1 we state some basic results about the general theory and the local theory.

The word "field" will mean "field of characteristic 2." When possible we use the notation and terminology of [**5**]. The matrix notation of [**7**] will also be employed.

1. **Preliminaries.** Let x be an anisotropic vector in V. The orthogonal transvection with respect to the anisotropic line Fx will be denoted by τ_x. Dieudonné has shown in [**4**] that each σ in $O(V)$ may be expressed as $\sigma=\tau_{x_1} \cdots \tau_{x_r}$ where $r \leqq \dim V=n$ and V is not a quaternary hyperbolic space over $\boldsymbol{F}_2$. In §1 we assume that $F\neq \boldsymbol{F}_2$ in order to avoid this exceptional case. In §2 the case $F=\boldsymbol{F}_2$ is precluded by the assumption that F is a local field.

The method of proof used to establish 43.6 in [**5**] may be employed to show that Ω is generated by elements of the type $\tau_x\tau_y\tau_x\tau_y$. Thus $\Omega \subseteq O^+$ and $\Omega \subseteq O'$.

The group O^+ may be characterized as the set of all elements of O which have an expression as a product of exactly n orthogonal transvections. Replacing the role of the discriminant by the Dickson invariant in the proof of 43.3b in [**5**] will establish that each σ in O^+ has an expression as a product of n orthogonal transvections where the first (or last) is chosen arbitrarily. In particular, any σ in O' or Ω has such an expression.

PROPOSITION 1. *Let V be a nondefective quadratic space of dimension n over the field F.*

(i) *If $n=2$ then $O'(V)=\Omega(V)$.*

(ii) *If U is a nondefective subspace of V, the groups $O'(U)$ and $\Omega(U)$*

Received by the editors November 29, 1968.

may be considered in a natural way as subgroups of $O'(V)$ *and* $\Omega(V)$ *respectively.*

(iii) *If* $\{u_1, \cdots, u_r\}$ *and* $\{v_1, \cdots, v_r\}$ *are sets of anisotropic vectors in* V *and* $\{Q(u_1), \cdots, Q(u_r)\}$ *is a permutation of* $\{Q(v_1), \cdots, Q(v_r)\}$ *then* $\tau_{u_1} \cdots \tau_{v_r} \in \tau_{v_1} \cdots \tau_{v_r} \Omega$.

(iv) *Let* $\sigma \in O'(V)$ *and assume that* $\sigma = \prod_{i=1}^{2r} \tau_{x_i}$. *If* V *contains a nondefective subspace* U *with* $\Omega(U) = O'(U)$ *and* $Q(x_i) \in Q(U)$, $1 \leqq i \leqq 2r$, *then* $\sigma \in \Omega(V)$.

PROOF. The proofs of these statements are the same as those used in [5] to prove the analogous results in the characteristic not 2 theory.

Now let F be a local field, $\mathfrak{o}$ the integers of F, $\mathfrak{u}$ the units of $\mathfrak{o}$, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, π a fixed prime element and $\overline{F}$ the residue class field of F. Since $\overline{F}$ is a finite field, the set $\{\alpha^2+\alpha \mid \alpha \in \overline{F}\} = \wp(\overline{F})$ is a subgroup of index 2 in the additive group of $\overline{F}$. We let 0 and ρ be representatives of $\overline{F}/\wp(\overline{F})$.

PROPOSITION 2. *Let* V *be a nondefective quadratic space over a local field* F.

(i) *If*

$$V \cong \begin{pmatrix} & 1 \\ a & a^{-1}\rho \end{pmatrix}$$

then $Q(V) = a\mathfrak{u}F^2$.

(ii) *If* V *is quaternary anisotropic then*

$$V \cong \begin{pmatrix} & 1 \\ 1 & \rho \end{pmatrix} \perp \begin{pmatrix} & 1 \\ \pi & \pi^{-1}\rho \end{pmatrix}$$

(iii) V *is universal if* $\dim V \geqq 4$ *and isotropic if* $\dim V \geqq 6$.

(iv) *If* V *is quaternary and* P *is binary with* $\Delta P \neq \Delta V$ *then* $P \rightarrowtail V$. *In particular if* P *and* V *are anisotropic then* $P \rightarrowtail V$.

PROOF. See [6].

2. The main result.

LEMMA 1. *Let* F *be a local field. Let* ϵ *be a fixed nonsquare unit. Then there exists a nondefective binary, anisotropic space over* F *which represents* $1, \pi, \pi\epsilon$ *and* ϵ.

PROOF.[1] The quadratic abelian extensions of F are characterized by the property that they are splitting fields of irreducible polynomials of the type X^2+X+d, $d \in F$. Let E be such an extension.

[1] The referee has provided an excellent modification of my original proof. I wish to acknowledge his suggestion and to thank him for it.

Then $E=F(\delta)$ where $\delta^2+\delta+d=0$. Consider the nondefective binary space over F given by the matrix

$$P\cong\begin{pmatrix} & 1\\ 1 & d\end{pmatrix}.$$

Clearly, P is anisotropic and $Q(P)=N_{E|F}(E)$. Thus we are done if we produce a quadratic abelian extension E over F whose norm group$N_{E|F}(\dot{E})$ contains the desired elements. By the existence theorem of local class field theory, see [2], it is enough to produce an open subgroup of index 2 in $\dot{F}$ which contains the elements 1, π, ϵ and $\pi\epsilon$. This we now do.

Let $u^{(n)}=\{\epsilon\in u\,|\,\epsilon\equiv 1 \bmod \pi^n\}$. Thus $u^{(0)}=u$ and $u^{(0)}\supset u^{(1)}\supset\cdots$. Now $u/u^{(1)}\cong\overline{F}^{\cdot}$ and $u^{(i)}/u^{(i+1)}\cong\overline{F}^{+}$ for $i\geqq 1$. Thus $u/u^{(4)}$ has order $(q-1)q^3$ where q is the cardinality of $\overline{F}$. Splitting $u/u^{(4)}$ into its unique p-primary components, we may write $u/u^{(4)}=A\oplus B$ where the order of $A=q-1$ and the order of $B=q^3$. Since q is a power of 2, the order of B is a power of 2 and at least 8. Since $u^{(1)}/u^{(4)}$ is a subgroup of $u/u^{(4)}$ of order q^3, $B=u^{(1)}/u^{(4)}$. Since $\lambda^4\in u^{(4)}$ for any $\lambda\in u^{(1)}$, B is not cyclic. Thus any element of B is contained in a subgroup C of index 2 in B.

Let ϵ be the nonsquare unit under consideration. The image of ϵ under the isomorphism from $u/u^{(4)}$ to $A\oplus B$ is contained in a subgroup $A\oplus C$ where C is of the type described above. Clearly $A\oplus C$ has index 2 in $A\oplus B$. Thus ϵ is contained in a subgroup G of u of index 2 with $G\supset u^{(4)}$. But then G is open in u and, since u is open in $\dot{F}$, G is open in $\dot{F}$. Denoting by $\langle\pi\rangle$ the cyclic subgroup of $\dot{F}$ generated by π, we see that $G\langle\pi\rangle=\bigcup_{n\in Z}G\pi^n$ is also open in $\dot{F}$ and $(\dot{F}:G\langle\pi\rangle)=(u\langle\pi\rangle:G\langle\pi\rangle)=2$. Thus $G\langle\pi\rangle$ is an open subgroup of $\dot{F}$ of index 2 that contains 1, π, ϵ and $\pi\epsilon$. Q.E.D.

THEOREM. *If V is a nondefective quadratic space over the local field F then $O'(V)=\Omega(V)$.*

PROOF. In light of 1(i), 2(iii) and the results of [3] we may assume that V is quaternary anisotropic. Hence $V=P_1\perp P_2$ where

$$P_1\cong\begin{pmatrix} & 1\\ 1 & \rho\end{pmatrix}\quad\text{and}\quad P_2\cong\begin{pmatrix} & 1\\ \pi & \pi^{-1}\rho\end{pmatrix}.$$

Let $\sigma\in O'(V)$. The universality of V and the results of §1 allow us to assume that $\sigma=\tau_{x_1}\tau_{x_2}\tau_{x_3}\tau_{x_4}$ where $Q(x_1)=1$. Since $\tau_x=\tau_{\alpha x}$, $\alpha\in\dot{F}$, we may assume that $Q(x_i)=\pi$ or $\pi\epsilon_i$, $\epsilon_i\in u$, $1\leqq i\leqq 4$. Since $\prod_{i=1}^4 Q(x_i)\in F^2$ we have two cases to consider:

(a) $Q(x_i)\in u$, $1\leqq i\leqq 4$.

In this case $Q(x_i)\in Q(P_1)$ by Proposition 2(i). Now we apply 1(iv) and 1(i).

(b) Exactly two of the $Q(x_i)$, $1\leqq i\leqq 4$, are of the form $Q(x_i)=\pi\epsilon_i$, $\epsilon_i\in u$. We may assume that $Q(x_2)=\epsilon_2$, $Q(x_3)=\pi\epsilon_3$ and $Q(x_4)=\pi\epsilon_4$ by Proposition 1(iii).

Since V is universal we may select y_1 and y_2 in V with $Q(y_1)=\pi$ and $Q(y_2)=\pi\epsilon_2$. Let $\Sigma=\tau_{y_1}\tau_{y_2}\tau_{x_3}\tau_{x_4}$. Clearly $\Sigma\in O'(V)$. Moreover, by applying 2(i), 1(i) and 1(iv) to P_2 we see that $\Sigma\in\Omega(V)$. If $\sigma\Sigma\in\Omega(V)$ we will be done. By applying 1(iii) we see that $\sigma\Sigma\in\Omega(V)$ if $\tau_{x_1}\tau_{x_2}\tau_{y_1}\tau_{y_2}\in\Omega(V)$. If $\epsilon_2\in u^2$ then $\tau_{x_1}\tau_{x_2}$ and $\tau_{y_1}\tau_{y_2}$ are in $O'(V)$. By applying 1(iv) to the spaces P_1 and P_2 respectively, we see that $\tau_{x_1}\tau_{x_2}$ and $\tau_{y_1}\tau_{y_2}$ are in $\Omega(V)$. Hence $\tau_{x_1}\tau_{x_2}\tau_{y_1}\tau_{y_2}\in\Omega(V)$. Thus we may assume that $\epsilon_2\notin u^2$. By the above lemma there is a nondefective, binary, anisotropic space P which represents 1, π, $\pi\epsilon_2$ and ϵ_2. We have $P\rightarrowtail V$ by 2(iv). Thus V contains a nondefective binary space B which represents 1, π, $\pi\epsilon_2$ and ϵ_2. Applying 1(iv) and 1(i) to B yields the desired result. Q.E.D.

References

1. C. Arf, *Untersuchungen über quadratische Formen in Körpern der Charakteristik* 2, J. Reine Angew. Math. **183** (1941), 148–167.

2. J. W. S. Cassels and A. Fröhlich, editors, *Algebraic number theory*, Thompson Book Co., Washington, D.C., 1967.

3. J. Dieudonné, *La géométrie des groupes classiques*, Springer-Verlag, Berlin, 1963.

4. ———, *Sur les générateurs des groupes classiques*, Summa Brasil. Math. **3** (1955), 149–178.

5. O. T. O'Meara, *Introduction to quadratic forms*, Springer-Verlag, Berlin, 1963.

6. C. R. Riehm, *Integral representation of quadratic forms in characteristic* 2, Amer. J. Math. **86** (1964), 25–62.

7. C. H. Sah, *Quadratic forms over fields of characteristic* 2, Amer. J. Math. **82** (1960), 812–830.

University of Notre Dame