EXTENSION OF OPERATOR VALUED SET FUNCTIONS WITH FINITE SEMIVARIATION

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I. **Introduction.** In recent work by R. J. Easton [3], D. J. Uherka [7], D. H. Tucker [3], [6], [7], and S. G. Wayment [8], questions dealing directly with the problem of extending a finitely additive operator valued set function with finite semivariation defined on a particular ring C of subsets of a compact Hausdorff space have been studied. The present note lends considerable support to the choice of this particular ring.

We consider a similar extension problem from a different viewpoint. In particular, we suppose only that C is a ring of subsets of a universal space T, each of E and F is a Banach space, $B(E, F)$ is the Banach space of all bounded linear operators $f: E \to F$, and $m: C \to B(E, F)$ is a finitely additive set function with finite semivariation. That is, if $A \in C$, we assume that $\sup \| \sum m(A_i) \cdot x_i \|$ is finite, where we take the supremum over all finite C-subdivisions $\{A_i\}$ of A and all elements x_i of unit norm in E. $H(C)$ is the hereditary σ-ring generated by C, i.e. $H(C)$ is the σ-ring consisting of all subsets of T which can be covered by countable unions from C. We use the semivariation $\tilde{m}$ of m to define an outer measure m^* on $H(C)$ in the obvious way, and we let $T(m)$ be the set of all elements A in $H(C)$ so that if $B \in H(C)$, then $m^*(B) = m^*(B \cap A) + m^*(B - A)$. It is shown in Chapter 12 of [5] and Chapter 1 of [1] that $T(m)$ is a σ-ring. $C(m)$ will be the largest class of subsets of T so that $T(m)$ forms an ideal in $C(m)$, i.e. the intersection of each element of $C(m)$ with any element in $T(m)$ lies in $T(m)$. And $\Sigma(m)$ will be the δ-ring of all elements in $C(m)$ with finite μ^* measure (Definition 3).

The main results of this note appear in Theorem 1 and Theorem 3. In Theorem 1 we show that vsr (Definition 1) $+ (C \subset T(m))$ implies that $\tilde{m} = |m|$. Furthermore, in case $C \subset T(m)$, we show that there is a unique extension $m_1: \Sigma(m) \to B(E, F)$ of m so that

(i) m_1 has finite semivariation $\tilde{m}_1$;

(ii) $\tilde{m}_1 = \mu^*$;

(iii) $\tilde{m}_1 = |m_1|$, the total variation of m_1;

(iv) $\tilde{m}_1$ extends $\tilde{m}$.

Thus we are able to conclude that $\tilde{m}$ and $\tilde{m}_1$ are countably additive on their domains. In Theorem 3 we show that essential one-dimen-

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sionality of the range of m is a necessary and sufficient condition for $\tilde{m}$ to be $|m|$. In addition an example is given in which m^* does not agree with $\tilde{m}$ on C, even though $C\subset T(m)$; and another example shows that C may be a very rich ring and $T(m)$ may be trivial.

DEFINITION 1. We say that a finitely additive set function $m: C \to B(E, F)$ is variationally semiregular at ϕ (vsr) provided that if $\{A_n\}$ is a decreasing sequence of sets in C whose intersection is empty and $\tilde{m}(A_1) < \infty$, then $\lim_n \tilde{m}(A_n) = 0$. Halmos says that $\tilde{m}$ is continuous from above at ϕ.

DEFINITION 2. Let C, $m: C\to B(E, F)$, and $H(C)$ be as before. Let $A\in H(C)$. Define $m^*(A)$ to be the inf $\Sigma\tilde{m}(A_n)$, where the infimum is taken over all countable C-coverings of A. Clearly m^* is an outer measure on $H(C)$.

DEFINITION 3. Let C, $m: C\to B(E, F)$, $H(C)$, $T(m)$, and $C(m)$ be as above. For $A\in C(m)$, define $\mu^*(A)$ to be sup $m^*(B)$, where the supremum is taken over all $B\in T(m)$ such that $B\subset A$.

EXAMPLE 1. Let E be the smallest ring containing the following subsets of $(0, 1]$: $\{\{\frac{1}{2}\}, (0, 1], (\frac{1}{2}, 1], (\frac{1}{3}, \frac{1}{2}], (\frac{1}{4}, \frac{1}{3}], \cdots\}$. For $e\in E$, define $K(e)$ to be 1 if e contains a set of the form $(0, a]$; otherwise, define $K(e)$ to be 0. It follows that K is additive on E. Certainly $H(E)$ is the power class P of $(0, 1]$; furthermore, if $A\in H(E)$, then $m^*(A)=0$. Hence $T(m)=\mathrm{P}$, and $C\subset T(m)$. However, it is clear that $\tilde{m}((0, 1])=1$, and thus $\tilde{m}\neq m^*$ on C. This example also shows that the additivity of $\tilde{m}$ need not imply countable subadditivity.

EXAMPLE 2. Let C denote the power class of $[0, 1]$. For $A\in C$, define $m(A)$ to be X_A, the characteristic function of A. We interpret X_A as being in the Banach space E of all bounded functions on $[0, 1]$ with the uniform norm, i.e. $E=B(S, E)$, where S is the scalar field. Then $H(C)=C$, $\tilde{m}(A)$ is 1 if A is nonempty, and $\tilde{m}(\phi)=0$. Similarly, $m^*(A)$ is 1 if A is nonempty, and $m^*(\phi)=0$. Therefore $T(m) = \{\phi, [0, 1]\}$, which forces m^* and μ^* to differ.

LEMMA 1. *Suppose that $m: C\to B(E, F)$ is a finitely additive set function with finite semivariation on C. Suppose furthermore that m is vsr and m^* is the outer measure on $H(C)$. Then $\tilde{m}=m^*$ on C.*

PROOF. It is clear in general that $m^*\leqq\tilde{m}$ on C. Suppose then that there is an $A\in C$ such that $m^*(A)<\tilde{m}(A)$. Then $m^*(A)<\infty$ and there is an $\epsilon>0$ and a disjoint sequence $\{A_n\}_{n=1}^{\infty}$ from C which covers A so that $m^*(A)+\epsilon<\tilde{m}(A)$ and $\sum_{n=1}^{\infty}\tilde{m}(A_n)-m^*(A)<\epsilon/2$. Let $B_n = A_n\cap A$ for each positive integer n. Thus $\sum_{n=1}^{\infty}\tilde{m}(B_n)-m^*(A)<\epsilon/2$, $B_n\in C$ for each n, and $\bigcup_n B_n=A$. Therefore $\tilde{m}(A)>\sum_{n=1}^{\infty}\tilde{m}(B_n)+\epsilon/2$. Let $U_k=\bigcup_{i=k}^{\infty}B_i$, and notice that for arbitrary k,

$$\tilde{m}(A) \leqq \sum_{i=1}^{k-1} \tilde{m}(B_i) + \tilde{m}(U_k).$$

Hence it follows that $\tilde{m}(U_k) > \epsilon/2$ for each k. But $U_k \in C$, $U_k \subset A$, and $\tilde{m}(A) < \infty$. Since $\bigcap_k U_k = \emptyset$ and m is vsr, we have a contradiction. Therefore we conclude that $m^* = \tilde{m}$ on C, and the lemma is proved.

For $A, B \in \Sigma(m)$, define $\rho(A, B)$ to be $\mu^*(A \triangle B)$. Then ρ defines a semimetric on $\Sigma(m)$.

We include a proof of the following lemma since we do not assume that m^* is $\tilde{m}$, and the analogous fact for total variation ($m^* = |m|$) is present in the form in which this result usually appears.

LEMMA 2. *If $C \subset T(m)$, then C is ρ-dense in $\Sigma(m)$.*

PROOF. Let $B \in \Sigma(m)$ and choose $\epsilon > 0$. Since $\mu^*(B)$ is finite, there is an element A in the σ-ring generated by C so that $A \subset B$ and $\mu^*(B - A) < \epsilon$. But $A \in H(C)$; hence, since $\mu^*(A)$ is finite, there is a disjoint sequence $\{R_i\}$ of elements from C covering A so that $\mu^*(A) + \epsilon > \sum_{i=1}^{\infty} \tilde{m}(R_i)$. Therefore, $\mu^*(A) + \epsilon > \sum_{i=1}^{\infty} m^*(R_i)$. Let M be a positive integer so that if $j \geqq M$ and $R^{(j)} = \bigcup_{i=1}^{j} R_i$, then $\left| \sum_{i=1}^{\infty} m^*(R_i) - m^*(R^{(j)}) \right| < \epsilon$. It follows that $\mu^*(B \triangle R^{(M)}) < 6\,\epsilon$, and the lemma is proved.

II. THEOREM 1. *Let C, m: $C \triangle B(E, F)$, $T(m)$, $\Sigma(m)$, $C(m)$, $H(C)$, m^*, and μ^* be as above, and consider the following conditions:*

(a) *F isomorphically isometric to C;*

(b) *$\tilde{m} = |m|$ on C;*

(c) *$\tilde{m}$ finitely additive in C;*

(d) *$C \subset T(m)$.*

Then (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d).

If m is vsr, then the following conditions are equivalent:

(c) *$\tilde{m}$ is finitely additive on C;*

(c^1) *$\tilde{m}$ is countably additive on C;*

(d) *$C \subset T(m)$;*

(e) *$C \subset T(m)$ and there exists a unique extension m_1: $\Sigma(m) \to B(E, F)$ of m such that m_1 satisfies conditions* (i)–(iv) *of the introduction.*

PROOF. Suppose that $\tilde{m}$ is finitely additive on C. Let $A \in C$ and suppose that $\{A_1, \cdots, A_n\}$ is a finite disjoint collection of sets in C whose union is A. Then $\sum_{i=1}^{n} \|m(A_i)\| \leqq \sum_{i=1}^{n} \tilde{m}(A_i) = \tilde{m}(A)$. Therefore $|m|(A) \leqq \tilde{m}(A)$, but the reverse inequality is clear, and $\tilde{m} = |m|$ on C.

Conversely, if $\tilde{m} = |m|$, then the additivity of $|m|$ on C implies that $\tilde{m}$ is additive on C. Therefore (b) and (c) are equivalent.

The argument that finite additivity of $\tilde{m}$ implies that $C\subset T(m)$ is well known, e.g. see [2, Volume 1, p. 135]. The hypothesis that the set function is countably additive is made, but finite additivity is all that is needed.

Finally, that (a) implies (b) is clear.

For the remainder of the proof, we suppose that m is vsr. From Lemma 1, we know that $m^*=\tilde{m}$. Hence, if $C\subset T(m)$, we conclude that $\tilde{m}$ is countably additive (and thus additive) on C. Conversely, if $\tilde{m}$ is additive on C, we have shown that $C\subset T(m)$. Since m^* is countably additive on $T(m)$, we conclude that $\tilde{m}$ is countably additive on C. Hence (c), (c^1), and (d) are shown to be equivalent.

Finally, let us suppose again that $C\subset T(m)$. We recall that $\Sigma(m)$ is the collection of all sets in $C(m)$ with finite μ^*-measure; hence $C\subset\Sigma(m)$. Notice that μ^* agrees with m^* on $T(m)$, a situation that did not exist in Example 2. From Lemma 2, we conclude that C is ρ-dense in $\Sigma(m)$. For $A\in\Sigma(m)$, let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets in C so that $\lim_n \rho(A_n, A)=0$. Then $\{m(A_n)\}_{n=1}^{\infty}$ defines a Cauchy sequence in $B(E, F)$, and we define $m_1(A)$ to be $\lim m(A_n)$. Since m is uniformly continuous with respect to ρ, it follows that m_1 is well defined, additive, extends m and has finite variation. Hence m_1 has finite semivariation.

The last assertion is that $\mu^*=\tilde{m}_1$ on $\Sigma(m)$. While the circumstances are somewhat different, the techniques employed here will carry over to that situation studied by Dinculeanu, providing a considerable simplification of a part of the proof of Theorem 3, p. 76 of [1].

It is a short exercise to prove that $\tilde{m}_1\leqq\mu^*$; thus it will suffice to prove that $\mu^*\leqq\tilde{m}_1$. Let $A\in\Sigma(m)$ and suppose that $\epsilon>0$. Since C is ρ-dense in $\Sigma(m)$, there is a set $B\in C$ so that $\mu^*(A-B)+\mu^*(B-A)<\epsilon$. We remark that $|\mu^*(B)-\tilde{m}(B)|<\epsilon$. Without loss of generality, suppose that $\tilde{m}_1(A)\geqq\tilde{m}_1(B)$. Then $\tilde{m}_1(A)-\tilde{m}_1(B)\leqq\tilde{m}_1(A\cap B)+\tilde{m}_1(A-B)-\tilde{m}_1(B)\leqq\rho(A, B)<\epsilon$. Hence $|\tilde{m}_1(A)-\tilde{m}_1(B)|<\epsilon$. Furthermore, $|\mu^*(A)-\tilde{m}(B)|<\epsilon$, and thus $\mu^*(A)<\mu^*(B)+\epsilon\leqq\tilde{m}_1(B)+\epsilon<\tilde{m}_1(A)+2\epsilon$. Therefore $\tilde{m}_1=\mu^*$ on $\Sigma(m)$, and it follows immediately that $\tilde{m}_1$ is countably additive on $\Sigma(m)$. Also, from an earlier part of the proof, we conclude that $\tilde{m}_1=|m_1|$ on $\Sigma(m)$. That $\tilde{m}_1=\mu^*$ on $\Sigma(m)$ immediately implies that $\tilde{m}_1$ extends $\tilde{m}$.

It is easy to see that any extension of m with property (ii) of the introduction is uniformly continuous with respect to ρ. Since C is ρ-dense in $\Sigma(m)$, it follows by this uniform continuity that (ii) forces uniqueness. The converse is clear since we have demonstrated one such extension that satisfies (ii), and the theorem is proved.

COROLLARY 1. (*Notation established in Chapter* 2 *of* [**1**].) *If any of the last four conditions of the theorem hold,* $U \leftrightarrow m$, $U_1 \leftrightarrow m_1$, *then* $\|U_A\| = |||U_A|||$ *for each* $A \in C$ *and* $\|U_{1B}\| = |||U_{1B}|||$ *for each* $B \in \Sigma(m)$.

In fact, we can say more. The corollary is equivalent to any of the last four conditions of the theorem.

COROLLARY 2. *Suppose that* m *is vsr,* $T \in C$, $\tilde{m}$ *is additive on* C, *and* $\mathfrak{T}$ *is the* σ*-ring generated by* C. *Then there is a unique extension* m_1 *of* m *to* $\mathfrak{T}$ *so that* $\tilde{m}_1 = m^*$.

A significant improvement in the theorem would be to reverse the implication between (c) and (d) without requiring vsr. However, this is not possible. Let E and K be as in Example 1. Let $F = \mathbf{C} \oplus \mathbf{C}$, equipped with the norm $\|(\alpha, \beta)\| = \max\{|\alpha|, |\beta|\}$. Define $m: E \to F = B(S, F)$ in the following way: if $\frac{1}{2} \in e$, define $m(e)$ to be $(K(e), \frac{1}{2})$; if $\frac{1}{2}$ is not in e, define $m(e)$ to be $(K(e), 0)$. Then m is additive on E and has finite semivariation. Furthermore, $C \subset T(m)$. However, $\tilde{m}$ is not additive. For clearly $A = (0, \frac{1}{2}] \in E$, $\tilde{m}(A) = 1$, $\tilde{m}((0, \frac{1}{3}]) = 1$, and $\tilde{m}((\frac{1}{3}, \frac{1}{2}]) = \frac{1}{2}$.

In view of an example given by D. H. Tucker [**6**] in which the semivariation of the entire space is finite but the total variation of any nontrivial interval of the ring under consideration is infinite, this theorem and the accompanying examples show that in general the construction found in most classical books does not carry over fruitfully to the semivariation setting.

THEOREM 2. *Suppose that* $C \subset T(m)$. *Then the following conditions are equivalent*:

(a) m *is vsr*;

(b) $\|m(A)\| \leqq m^*(A)$ *for each* $A \in C$;

(c) $\tilde{m}$ *is countably additive on* C;

(d) $\tilde{m} = m^*$ *on* C.

PROOF. The plan of attack is to show that (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(a). That (a)$\Rightarrow$(b) is clear since from Lemma 1 we know that $\tilde{m} = m^*$ on C, and certainly $\|m(A)\| \leqq \tilde{m}(A)$ for each $A \in C$. Now (b) implies that $|m| \leqq m^*$, but $\tilde{m} \leqq |m| \leqq m^*$ and m^* is countably additive on C. Therefore $\tilde{m}$ must also be countably additive. It follows immediately that (c)$\Rightarrow$(d). For the countable additivity of $\tilde{m}$ implies that $\tilde{m}$ is outer regular on C, and by Lemma 1 we see that $\tilde{m} = m^*$. The final implication is also clear. Since $\tilde{m} = m^*$ and m^* is outer regular (for example, see [**5**, p. 51]), it follows that $\tilde{m}$ is outer regular—a condition that implies vsr.

Perhaps it should be stated explicitly that under the equivalent conditions of this theorem, we have that $m^* = \tilde{m} = |m|$ on C.

The preceding remarks are somewhat suggestive of the scalar case. This connection will be made clearer in Theorem 3. But first we make some additional observations related to the preceding theorems and examples.

1. If $m: C \to B(E, F)$ has finite semivariation and is vsr, then m is outer regular.

2. Suppose that $m: C \to B(E, F)$ has finite semivariation, m is vsr, and $C \subset T(m)$. If $U: M_E(C) \to F$ is in correspondence with m, then there is a unique extension $U_1: M_E(\Sigma(m)) \to F$ of U so that $\|U_1(B)\| \leqq \mu^*(B)$ for each $B \in \Sigma(m)$. Furthermore, if there is only one extension U_1 of U, then $\|U_1(B)\| \leqq \mu^*(B)$ for $B \in \Sigma(m)$.

3. It follows from Example 1 that the density of C in $\Sigma(m)$ does not imply that m is continuous with respect to ρ. Hence this extension process cannot be carried out in general.

4. Example 2 shows that the norm of m being bounded by m^* does not imply that $C \subset T(m)$.

Now suppose that $A \in C$ and let $\mathfrak{A}_A^C$ denote the class of all C-subdivisions σ: $\{A_1, \cdots, A_n\}$ of A. Then $\mathfrak{A}_A^C$ forms a directed family via the following direction: if $\alpha, \beta \in \mathfrak{A}_A^C$, then $\alpha \leqq \beta$ iff α refines β. For $\alpha \in \mathfrak{A}_A^C$, define $m(\alpha)$ to be $\sum_\alpha \|m(A_i)\|$. For a fixed $A \in C$ and $w \in F^*$, we view $m(A)^* w$ as an element in E^*, i.e. if $x \in E$, $m(A)$ carries x into F and in turn w carries $m(A) \cdot x$ into the scalar field. And we define $m_w(A)$ to be $m(A)^* w$. Therefore $\bar{m}_w$ will simply denote the total variation of the operator m_w.

THEOREM 3. *Let $\tilde{m}$ denote the semivariation and $|m|$ the total variation of $m: C \to B(E, F)$. If $A \in C$, then $\tilde{m}(A) = |m|(A)$ iff* (*) *there exists an $\alpha \in \mathfrak{A}_A^C$ and a $w \in F^*(\|w\| \leqq 1)$ so that if $\beta \in \mathfrak{A}_A^C$ and $\beta \leqq \alpha$, then the norms of the components of the sum $m(\beta)$ are* (*approximately*) *singly generated by w, i.e. $\beta \leqq \alpha \Rightarrow m(\beta)$ and $\sum_\beta \|m(A_i)^* w\| = m_w(\beta)$ are close.*

PROOF. In order to simplify the notation, we suppose that $|m|(A) < \infty$. The context makes it clear how to proceed otherwise. Suppose that (*) is true. If $\epsilon > 0$, then there is a $\sigma \in \mathfrak{A}_A^C$ $(\sigma \leqq \alpha)$ so that if $\beta \leqq \sigma$, then $|m(\beta) - |m|(A)| < \epsilon$. Choose one such β. Then $|m|(A) < m_w(\beta) + 2\epsilon \leqq \bar{m}_w(A) + 2\epsilon$. Thus

$$|m|(A) \leqq \sup_{w \in F^*; \|w\| \leqq 1} \bar{m}_w(A) \Rightarrow |m|(A) = \tilde{m}(A).$$

Conversely, suppose that $|m|(A) = \tilde{m}(A)$. Then there is a $w \in F^*$, $\|w\| \leqq 1$, so that $||m|(A) - \bar{m}_w(A)| < \epsilon$. Hence there is a $\sigma \in \mathfrak{A}_A^C$ so that if $\beta \leqq \sigma$, then $\left| m(\beta) - |m|(A) \right| < \epsilon$ and $\left| \bar{m}_w(A) - m_w(\beta) \right| < \epsilon$, and the theorem is proved.

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