HYPERBOLIC SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACE

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In [1] Professor Kobayashi constructed an invariant pseudo-distance $d_M$ on each complex manifold $M$. If the pseudo-distance $d_M$ is a true distance, the complex manifold is said to be hyperbolic. It is known (see [1]) that if $M$ admits a hermitian metric of strongly negative curvature then $M$ is hyperbolic. In this paper, examples of hyperbolic manifolds are obtained by a more elementary method. In the process, it shall be shown that any covering manifold of the complement of any $2n$ hyperplanes in general position in $P_n(C)$ is not biholomorphically equivalent to a bounded domain of $C^n$. This gives a negative answer to a question posed by Professor Chern.

Before proceeding, it will be useful to state some of Kobayashi’s basic results concerning the pseudo-distance $d_M$. For further details and proofs, see [1].

**Theorem 1.** If $f: M \to N$ is holomorphic, then $f$ is distance decreasing with respect to $d_M$ and $d_N$. Thus if $N$ is hyperbolic and $d_M=0$, then $f$ is constant. If $N$ is hyperbolic and there exists a 1-1 holomorphic mapping $f: M \to N$, then $M$ is hyperbolic.

**Theorem 2.** Let $\tilde{M}$ be a covering manifold of $M$. Then $\tilde{M}$ is hyperbolic if and only if $M$ is hyperbolic.

**Theorem 3.** If $M$ and $N$ are hyperbolic, then $M \times N$ is hyperbolic.

**Theorem 4.** $D^n$, the unit disc in $C^n$, is hyperbolic. $C-\{0, 1\}$ is hyperbolic. $C$ and $C-\{0\}$ are not hyperbolic and, in fact, $d_C=0=d_{C-\{0\}}$.

The following notation is necessary in order to give the examples. Let

$K_n = \{a=a_1 \cup a_2 \cup a_3 \text{ where the } a_i \text{ are distinct hyperplanes in } P_n(C) \text{ which intersect in an } n-2 \text{ dimensional subspace of } P_n(C)\}$.

$L_n = \{A = \sigma_1 \cup \cdots \cup \sigma_n \mid \sigma_\alpha = a_{\alpha 1} \cup a_{\alpha 2} \cup a_{\alpha 3} \in K_n \text{ and the following conditions are satisfied:}\}$

(i) For each $1 \leq \alpha \leq n$ there exists $\beta < \alpha$ and there exists $(i, k)$ such that $a_{\alpha i} = a_{\beta k}$. Furthermore the $i$ is unique.

(ii) If $b_\alpha = a_{\alpha 1} \cap a_{\alpha 2} \cap a_{\alpha 3}$, then for any set of $n$ hyperplanes $\{y_1, \cdots, y_n\}$ with $b_{\alpha} \subset y_\alpha \subset A$, we have $y_1 \cap \cdots \cap y_n = \text{point}$.

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Remarks. A member of $L_n$ is a union of $n$ elements $\sigma_1, \ldots, \sigma_n$ of $K_n$ which satisfy two conditions. The first condition says that exactly one of the hyperplanes of $\sigma_\alpha$ is in a previous $\sigma_\beta$. Thus an element of $L_n$ is the union of $2n+1$ hyperplanes. The second condition forces the $2n+1$ hyperplanes to have a nice relative position. For $n=1$ or 2, the first condition implies the second.

Theorem 5. Let $A \in L_n$. Then $M = P_n(C) - A$ is hyperbolic.

Proof. Let $p \in M$ and $q \in M$ be such that $d_M(p, q) = 0$. Let $A = \sigma_1 \cup \cdots \cup \sigma_n$, $\sigma_\alpha = a_\alpha \cup a_\alpha \cup a_\alpha$, and $b_\alpha = a_\alpha \cap a_\alpha \cap a_\alpha$. Let $N_\alpha = P_n(C) - \sigma_\alpha$. We can choose coordinates in $N_\alpha$ such that

$$N_\alpha = C^n - \{(z_1, \ldots, z_n) \mid z_n = 0 \text{ or } z_n = 1\}.$$ 

Define $\phi_\alpha : N_\alpha \to C - \{0, 1\}$ by $\phi_\alpha(z_1, \ldots, z_n) = z_n$. Since $C - \{0, 1\}$ is hyperbolic, this says that $\phi_\alpha(p) = \phi_\alpha(q)$. Thus there exists a hyperplane $y_\alpha$ such that \{p, q\} $\subset y_\alpha$ and $b_\alpha \subset y_\alpha \subset A$. Doing this for all $\alpha$, we have \{p, q\} $\subset y_\alpha \cap \cdots \cap y_\alpha$. By property (ii) in the definition of $L_n$, we have $p = q$ and therefore $M$ is hyperbolic. This completes the proof.

The previous theorem shows that if $A$ is the union of $2n+1$ hyperplanes in $P_n(C)$ which have the proper relative position, then $P_n(C) - A$ is hyperbolic. If $A = \sigma_1 \cup \cdots \cup \sigma_n \in L_n$ has the property that one hyperplane, say $a_{11}$, is common to all $\sigma_\alpha$, then $P_n(C) - A$ is equivalent to $C - \{0, 1\} \times \cdots \times C - \{0, 1\}$. In this case, $A$ is given in homogeneous coordinates by the equation

$$z_0z_1 \cdots z_n(z_0 - z_n)(z_1 - z_n) \cdots (z_{n-1} - z_n) = 0.$$ 

For $n = 1$ or $n = 2$, this is the only example we obtain. However, for $n \geq 3$ we obtain more. For example, each of the following equations define an element of $L_1$.

$$z_0 \cdots z_4(z_0 - z_4)(z_1 - z_4)(z_2 - z_4)(z_3 - z_4) = 0,$$

$$z_0 \cdots z_4(z_0 - z_4)(z_1 - z_4)(z_2 - z_4)(z_2 - z_3) = 0,$$

$$z_0 \cdots z_4(z_0 - z_4)(z_1 - z_4)(z_1 - z_2)(z_2 - z_3) = 0.$$ 

We now consider the complement of $2n$ hyperplanes in $P_n(C)$. Let $V = \sigma_1 \cup \cdots \cup \sigma_{2n}$ be the union of $2n$ hyperplanes in $P_n(C)$. We say that $V$ satisfies property P if (after reordering the $a_i$ if necessary):

(P) There exists points $p$ and $q$ and some $0 \leq k \leq 2n$ such that $p \in a_1 \cap \cdots \cap a_k$ and $q \in a_{k+1} \cap \cdots \cap a_{2n}$, and such that the line determined by $p$ and $q$ is not contained in any of the hyperplanes $a_i$. 

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Theorem 6. If $V$ is the union of $2n$ hyperplanes in $P_n(C)$ and $V$ satisfies property P, then $P_n(C) - V$ is not hyperbolic.

Proof. Property P implies that there exists a nonconstant holomorphic map of $C - \{0\}$ into $P_n(C) - V$. Thus by Theorems 1 and 4, $P_n(C) - V$ is not hyperbolic.

Corollary 1. If $V$ is the union of $2n$ hyperplanes in general position in $P_n(C)$, then $P_n(C) - V$ is not hyperbolic.

Proof. Let $V = a_1 \cup \cdots \cup a_{2n}$, $p \in a_1 \cap \cdots \cap a_n$ and $q \in a_{n+1} \cap \cdots \cap a_{2n}$. If the line determined by $p$ and $q$ is contained in $a_k$, then $a_1 \cap \cdots \cap a_n \cap a_k \neq \emptyset$ and $a_k \cap a_{n+1} \cap \cdots \cap a_{2n} \neq \emptyset$. This is impossible since the $a_i$ are in general position. Thus $V$ satisfies property P.

Corollary 2. If $n \leq 5$ and $V$ is the union of any $2n$ hyperplanes in $P_n(C)$, then $P_n(C) - V$ is not hyperbolic.

Proof. This is proved by considering the different possible ways in which the hyperplanes could intersect. For example, if $n = 2$ and $V = a_1 \cup \cdots \cup a_4$, there are three cases (up to relabelling the $a_i$):

1. $a_1 \cap \cdots \cap a_4 \neq \emptyset$,
2. $a_1 \cap \cdots \cap a_4 = \emptyset$ and $a_1 \cap a_2 \cap a_3 \neq \emptyset$,
3. the $a_i$ are in general position.

In each case it is easy to show that $V$ satisfies property P. Similar arguments work for $n = 3, 4$ or 5.

Remark. I feel that if $V$ is the union of any $2n$ hyperplanes in $P_n(C)$, then $V$ satisfies property P. This would imply that a minimum of $2n + 1$ hyperplanes must be removed from $P_n(C)$ in order to obtain a hyperbolic space. However, the arguments used to prove this for $n \leq 5$ do not seem to generalize.

Corollary 3. Let $V$ be as in Theorem 6 and let $\bar{M}$ be a covering manifold of $P_n(C) - V$. Then any bounded holomorphic map $f: \bar{M} \to C$ is not 1-1. In particular, $\bar{M}$ is not biholomorphically equivalent to a bounded domain of $C^n$.

Proof. Any bounded domain is hyperbolic. Theorems 2 and 6 imply that $\bar{M}$ is not hyperbolic. Therefore Theorem 1 implies that $f$ is not 1-1. This completes the proof.

Let $V$ be a complete quadrilateral in $P_2(C)$ with diagonal $T$. Then Corollary 2 says that $M = P_2(C) - V$ is not hyperbolic and Corollary 3 says that any covering $\bar{M}$ of $M$ is not biholomorphically equivalent.
to a bounded domain in \( C \). However, \( V \cup T \subseteq \mathbb{L}_2 \) and therefore \( P_2(C) - V \cup T \) is equivalent to \( C - \{0, 1\} \times C - \{0, 1\} \), which is covered by \( D' \times D' \). These last results have been obtained independently by Wilhelm Stoll [2].

We finish with an example. Let \( n \geq 2 \) and let \( A^d \subset P_n(C) \) be the variety defined by the homogeneous equation \( z_0^d + \cdots + z_n^d = 0 \) where \( d \) is a positive integer. Then \( M^d = P_n(C) - A^d \) is not hyperbolic for any \( d \). To see this, let \( U_n \) be the coordinate neighborhood obtained by setting \( z_n = 1 \). Then the map \( f : C \to U_n \) defined by \( f(z) = (z, (-1)^{1/d}z, 0, \cdots, 0) \) is nonconstant. Therefore \( M^d \) is not hyperbolic.

References


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