HYPERBOLIC SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACE

PETER KIERNAN

In [1] Professor Kobayashi constructed an invariant pseudo-distance d_M on each complex manifold M. If the pseudo-distance d_M is a true distance, the complex manifold is said to be hyperbolic. It is known (see [1]) that if M admits a hermitian metric of strongly negative curvature then M is hyperbolic. In this paper, examples of hyperbolic manifolds are obtained by a more elementary method. In the process, it shall be shown that any covering manifold of the complement of any $2n$ hyperplanes in general position in $P_n(C)$ is not biholomorphically equivalent to a bounded domain of C^n. This gives a negative answer to a question posed by Professor Chern.

Before proceeding, it will be useful to state some of Kobayashi's basic results concerning the pseudo-distance d_M. For further details and proofs, see [1].

THEOREM 1. *If $f: M \to N$ is holomorphic, then f is distance decreasing with respect to d_M and d_N. Thus if N is hyperbolic and $d_M \equiv 0$, then f is constant. If N is hyperbolic and there exists a* 1-1 *holomorphic mapping $f: M \to N$, then M is hyperbolic.*

THEOREM 2. *Let $\tilde{M}$ be a covering manifold of M. Then $\tilde{M}$ is hyperbolic if and only if M is hyperbolic.*

THEOREM 3. *If M and N are hyperbolic, then $M \times N$ is hyperbolic.*

THEOREM 4. *D^n, the unit disc in C^n, is hyperbolic. $C - \{0, 1\}$ is hyperbolic. C and $C - \{0\}$ are not hyperbolic and, in fact, $d_C \equiv 0 \equiv d_{C-\{0\}}$.*

The following notation is necessary in order to give the examples. Let

$K_n = \{\sigma = a_1 \cup a_2 \cup a_3$ where the a_i are distinct hyperplanes in $P_n(C)$ which intersect in an n-2 dimensional subspace of $P_n(C)\}$.

$L_n = \{A = \sigma_1 \cup \cdots \cup \sigma_n \mid \sigma_\alpha = a_{\alpha 1} \cup a_{\alpha 2} \cup a_{\alpha 3} \in K_n$ and the following conditions are satisfied:

(i) For each $1 < \alpha \leqq n$ there exists $\beta < \alpha$ and there exists (i, k) such that $a_{\alpha i} = a_{\beta k}$. Furthermore the i is unique.

(ii) If $b_\alpha = a_{\alpha 1} \cap a_{\alpha 2} \cap a_{\alpha 3}$, then for any set of n hyperplanes $\{y_1, \cdots, y_n\}$ with $b_\alpha \subset y_\alpha \not\subset A$, we have $y_1 \cap \cdots \cap y_n = \text{point}\}$.

Received by the editors January 9, 1969.

REMARKS. A member of L_n is a union of n elements $\sigma_1, \cdots, \sigma_n$ of K_n which satisfy two conditions. The first condition says that exactly one of the hyperplanes of σ_α is in a previous σ_β. Thus an element of L_n is the union of $2n+1$ hyperplanes. The second condition forces the $2n+1$ hyperplanes to have a nice relative position. For $n=1$ or 2, the first condition implies the second.

THEOREM 5. *Let $A \in L_n$. Then $M = P_n(C) - A$ is hyperbolic.*

PROOF. Let $p \in M$ and $q \in M$ be such that $d_M(p, q) = 0$. Let $A = \sigma_1 \cup \cdots \cup \sigma_n$, $\sigma_\alpha = a_{\alpha 1} \cup a_{\alpha 2} \cup a_{\alpha 3}$ and $b_\alpha = a_{\alpha 1} \cap a_{\alpha 2} \cap a_{\alpha 3}$. Let $N_\alpha = P_n(C) - \sigma_\alpha$. We can choose coordinates in N_α such that

$$N_\alpha = C^n - \{(z_1, \cdots, z_n) \mid z_n = 0 \text{ or } z_n = 1\}.$$

Define $\phi_\alpha : N_\alpha \to C - \{0, 1\}$ by $\phi_\alpha(z_1, \cdots, z_n) = z_n$. Since $C - \{0, 1\}$ is hyperbolic, this says that $\phi_\alpha(p) = \phi_\alpha(q)$. Thus there exists a hyperplane y_α such that $\{p, q\} \subset y_\alpha$ and $b_\alpha \subset y_\alpha \not\subset A$. Doing this for all α, we have $\{p, q\} \subset y_1 \cap \cdots \cap y_n$. By property (ii) in the definition of L_n, we have $p = q$ and therefore M is hyperbolic. This completes the proof.

The previous theorem shows that if A is the union of $2n+1$ hyperplanes in $P_n(C)$ which have the proper relative position, then $P_n(C) - A$ is hyperbolic. If $A = \sigma_1 \cup \cdots \cup \sigma_n \in L_n$ has the property that one hyperplane, say a_{11}, is common to all σ_α, then $P_n(C) - A$ is equivalent to $C - \{0, 1\} \times \cdots \times C - \{0, 1\}$. In this case, A is given in homogeneous coordinates by the equation

$$z_0 z_1 \cdots z_n (z_0 - z_n)(z_1 - z_n) \cdots (z_{n-1} - z_n) = 0.$$

For $n=1$ or $n=2$, this is the only example we obtain. However, for $n \geq 3$ we obtain more. For example, each of the following equations define an element of L_4.

$$z_0 \cdots z_4 (z_0 - z_4)(z_1 - z_4)(z_2 - z_4)(z_3 - z_4) = 0,$$

$$z_0 \cdots z_4 (z_0 - z_4)(z_1 - z_4)(z_2 - z_4)(z_2 - z_3) = 0,$$

$$z_0 \cdots z_4 (z_0 - z_4)(z_1 - z_4)(z_1 - z_2)(z_2 - z_3) = 0.$$

We now consider the complement of $2n$ hyperplanes in $P_n(C)$. Let $V = a_1 \cup \cdots \cup a_{2n}$ be the union of $2n$ hyperplanes in $P_n(C)$. We say that V satisfies property P if (after reordering the a_i if necessary):

(P) There exists points p and q and some $0 \leq k \leq 2n$ such that $p \in a_1 \cap \cdots \cap a_k$ and $q \in a_{k+1} \cap \cdots \cap a_{2n}$, and such that the line determined by p and q is not contained in any of the hyperplanes a_j.

THEOREM 6. *If V is the union of $2n$ hyperplanes in $P_n(C)$ and V satisfies property* P, *then $P_n(C)-V$ is not hyperbolic.*

PROOF. Property P implies that there exists a nonconstant holomorphic map of $C-\{0\}$ into $P_n(C)-V$. Thus by Theorems 1 and 4, $P_n(C)-V$ is not hyperbolic.

COROLLARY 1. *If V is the union of $2n$ hyperplanes in general position in $P_n(C)$, then $P_n(C)-V$ is not hyperbolic.*

PROOF. Let $V=a_1\cup\cdots\cup a_{2n}$, $p\in a_1\cap\cdots\cap a_n$ and $q\in a_{n+1}\cap\cdots\cap a_{2n}$. If the line determined by p and q is contained in a_k, then $a_1\cap\cdots\cap a_n\cap a_k\neq\emptyset$ and $a_k\cap a_{n+1}\cap\cdots\cap a_{2n}\neq\emptyset$. This is impossible since the a_i are in general position. Thus V satisfies property P.

COROLLARY 2. *If $n\leqq 5$ and V is the union of any $2n$ hyperplanes in $P_n(C)$, then $P_n(C)-V$ is not hyperbolic.*

PROOF. This is proved by considering the different possible ways in which the hyperplanes could intersect. For example, if $n=2$ and $V=a_1\cup\cdots\cup a_4$, there are three cases (up to relabelling the a_i):

(1) $a_1\cap\cdots\cap a_4\neq\emptyset$,

(2) $a_1\cap\cdots\cap a_4=\emptyset$ and $a_1\cap a_2\cap a_3\neq\emptyset$,

(3) the a_i are in general position.

In each case it is easy to show that V satisfies property P. Similar arguments work for $n=3$, 4 or 5.

REMARK. I feel that if V is the union of any $2n$ hyperplanes in $P_n(C)$, then V satisfies property P. This would imply that a minimum of $2n+1$ hyperplanes must be removed from $P_n(C)$ in order to obtain a hyperbolic space. However, the arguments used to prove this for $n\leqq 5$ do not seem to generalize.

COROLLARY 3. *Let V be as in Theorem* 6 *and let $\tilde{M}$ be a covering manifold of $P_n(C)-V$. Then any bounded holomorphic map $f:\tilde{M}\rightarrow C$ is not* 1-1. *In particular, $\tilde{M}$ is not biholomorphically equivalent to a bounded domain of C^n.*

PROOF. Any bounded domain is hyperbolic. Theorems 2 and 6 imply that $\tilde{M}$ is not hyperbolic. Therefore Theorem 1 implies that f is not 1-1. This completes the proof.

Let V be a complete quadrilateral in $P_2(C)$ with diagonal T. Then Corollary 2 says that $M=P_2(C)-V$ is not hyperbolic and Corollary 3 says that any covering $\tilde{M}$ of M is not biholomorphically equivalent to a bounded domain in C. However, $V\cup T\in L_2$ and therefore $P_2(C)-V\cup T$ is equivalent to $C-\{0, 1\}\times C-\{0, 1\}$, which is covered by $D'\times D'$. These last results have been obtained independently by Wilhelm Stoll [2].

We finish with an example. Let $n\geqq 2$ and let $A^d\subset P_n(C)$ be the variety defined by the homogeneous equation $z_0^d+\cdots+z_n^d=0$ where d is a positive integer. Then $M^d=P_n(C)-A^d$ is not hyperbolic for any d. To see this, let U_n be the coordinate neighborhood obtained by setting $z_n=1$. Then the map $f: C\rightarrow U_n$ defined by $f(z)=(z, (-1)^{1/d}z, 0, \cdots, 0)$ is nonconstant. Therefore M^d is not hyperbolic.

References

1. Shoshichi Kobayashi, *Invariant distances on complex manifolds and holomorphic mappings*, J. Math. Soc. Japan **19** (1967), 460–480.

2. Wilhelm Stoll, *About the universal covering of the complement of a complete quadrilateral*, Proc. Amer. Math. Soc. **22** (1969), 326–327.

University of California, Berkeley