FUNCTIONS POSITIVE DEFINITE IN $C[0, 1]$[1]

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A series of papers by I. J. Schoenberg discusses positive definite functions and isometries between metric spaces. In particular, we have the following definition from [3]. Let S be a metric space with the distance between points P and Q denoted by PQ. A real function ϕ, defined on the range of values of the distances, is said to be positive definite in S if

$$\sum_{i,j=1}^{n} \phi(P_iP_j)\rho_i\rho_j \geqq 0 \tag{1}$$

for arbitrary real ρ_i, $n=2, 3, \cdots$, and any n points P_i of S. If, in particular, S is the space $C\,[0, 1]$, we have the following

THEOREM. *Let $f(x)$ be positive definite in $C[0, 1]$. Then*

$$f(x) = k \geqq 0 \qquad \text{if } x > 0, \quad \text{and} \quad f(0) \geqq k. \tag{2}$$

Conversely, every $f(x)$ defined by (2) *is positive definite in $C[0, 1]$.*

The proof depends on a

LEMMA. *Let $0<x\leqq y\leqq 2x$. If the set of all pairs (i, j) of positive integers such that $1\leqq i<j\leqq n$ is partitioned into two sets A and B, then it is possible to find functions P_j in $C[0, 1]$ such that*

$$\|P_i - P_j\| = x, \quad (i, j) \in A, \qquad \|P_i - P_j\| = y, \quad (i, j) \in B.$$

PROOF. The lemma follows directly from a theorem of Banach and Mazur [1, p. 187], which states that every separable metric space may be isometrically imbedded in $C[0, 1]$.

PROOF OF THEOREM. Let ϕ be positive definite in $C[0, 1]$, with $\phi(x)=a$, $\phi(y)=b$, and choose P's as in the lemma. From (1) we get

$$\{\phi(0) - b\} \sum \rho_j^2 + b\{\sum \rho_j\}^2 + 2(a - b)\sum_A \rho_i\rho_j \geqq 0, \tag{3}$$

and similarly with interchanges of a with b and A with B.

Suitable choices of ρ_j, A, and n in (3) permit the following deductions:

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[1] This result was previously reported in [2]. The present version responds to a request for publication of a brief proof. A more leisurely discussion with more references is available from the author.

(i) Let $\sum \rho_j = 0$ and A be empty. Then $\phi(0) \geqq b$.

(ii) Let $n=2m$ with m of the ρ_j's $=-1$, the rest $=+1$. Let $A = \{(i,j) \mid \rho_i\rho_j = -1\}$ and let m become large. Then $a-b \leqq 0$. Similarly, $b-a \leqq 0$, so $a=b$.

(iii) Let all $\rho_j = 1$ and let n become large. Then $a \geqq 0$.

Steps (ii) and (iii) show that ϕ is constant on any interval $[x, 2x]$ for positive x, whence it follows that it is constant for all positive x. The theorem is thus proved except for the converse, which is trivial.

References

1. S. Banach, *Théorie des opérations linéaires*, Chelsea, New York, 1955.

2. S. J. Einhorn, *Functions positive definite in the space* C, Amer. Math. Monthly (4) **75** (1968), 393.

3. I. J. Schoenberg, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. **44** (1938), 522–536.

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