A series of papers by I. J. Schoenberg discusses positive definite functions and isometries between metric spaces. In particular, we have the following definition from [3]. Let \( S \) be a metric space with the distance between points \( P \) and \( Q \) denoted by \( PQ \). A real function \( \phi \), defined on the range of values of the distances, is said to be positive definite in \( S \) if

\[
\sum_{i,j=1}^{n} \phi(P_iP_j)\rho_i\rho_j \geq 0
\]

for arbitrary real \( \rho_i, \ n = 2, 3, \ldots \), and any \( n \) points \( P_i \) of \( S \). If, in particular, \( S \) is the space \( C[0, 1] \), we have the following

**Theorem.** Let \( f(x) \) be positive definite in \( C[0, 1] \). Then

\[
f(x) = k \geq 0 \quad \text{if} \quad x > 0, \quad \text{and} \quad f(0) \geq k.
\]

Conversely, every \( f(x) \) defined by (2) is positive definite in \( C[0, 1] \).

The proof depends on a

**Lemma.** Let \( 0 < x \leq y \leq 2x \). If the set of all pairs \((i, j)\) of positive integers such that \( 1 \leq i < j \leq n \) is partitioned into two sets \( A \) and \( B \), then it is possible to find functions \( P_i \) in \( C[0, 1] \) such that

\[
\|P_i - P_j\| = x, \quad (i, j) \in A, \quad \|P_i - P_j\| = y, \quad (i, j) \in B.
\]

**Proof.** The lemma follows directly from a theorem of Banach and Mazur [1, p. 187], which states that every separable metric space may be isometrically imbedded in \( C[0, 1] \).

**Proof of Theorem.** Let \( \phi \) be positive definite in \( C[0, 1] \), with \( \phi(x) = a, \phi(y) = b \), and choose \( P_i \)'s as in the lemma. From (1) we get

\[
\{ \phi(0) - b \} \sum \rho_i^2 + b \left\{ \sum \rho_i \right\}^2 + 2(a - b) \sum_A \rho_i \rho_j \geq 0,
\]

and similarly with interchanges of \( a \) with \( b \) and \( A \) with \( B \).

Suitable choices of \( \rho_i, \ A, \) and \( n \) in (3) permit the following deductions:

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1 This result was previously reported in [2]. The present version responds to a request for publication of a brief proof. A more leisurely discussion with more references is available from the author.
(i) Let $\sum \rho_j = 0$ and $A$ be empty. Then $\phi(0) \geq b$.

(ii) Let $n = 2m$ with $m$ of the $\rho_j$'s $= -1$, the rest $= +1$. Let $A = \{(i, j) | \rho_j = -1\}$ and let $m$ become large. Then $a - b \leq 0$. Similarly, $b - a \leq 0$, so $a = b$.

(iii) Let all $\rho_j = 1$ and let $n$ become large. Then $a \geq 0$.

Steps (ii) and (iii) show that $\phi$ is constant on any interval $[x, 2x]$ for positive $x$, whence it follows that it is constant for all positive $x$. The theorem is thus proved except for the converse, which is trivial.

References
