ISOMETRIES BETWEEN B^*-ALGEBRAS

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In [2], Kadison proves the following theorem:

Let A and B be C^*-algebras each of which contains an identity. Then if T is a linear isometry mapping A onto B, there exists a C^*-isomorphism τ mapping A onto B (i.e. τ is a linear isomorphism which preserves selfadjoints and power structure) and a unitary element $v\in B$ such that $T=v\tau$.

In recent years, the theory of numerical range, developed in [3], has provided techniques which have considerably simplified the proofs of certain results in the theory of B^*-algebras. The following question, posed by G. Lumer at the North British Functional Analysis Seminar held at Edinburgh in April 1968, is, therefore, natural: Can one prove the above theorem of Kadison using the techniques of the theory of numerical range? Lumer showed that such a proof can be given when the algebras concerned are commutative.

In this paper, we give a simple, intrinsic proof of Kadison's result, using certain elementary notions from the theory of numerical range.

We note that if A is a B^*-algebra with identity 1, the set

$$H = \{x \in A: \|1 + i\alpha x\| \leqq 1 + o(\alpha), \alpha \in \boldsymbol{R}, \alpha \to 0\}$$

coincides with the set of selfadjoint elements of A. This is proved by [3, Theorem 21].

In the sequel, A and B are B^*-algebras, each containing an identity 1, and T is a linear isometry mapping A onto B. A_1 and $H(A)$ denote respectively the closed unit ball and the set of hermitian elements of A. A' denotes the space of continuous linear functionals on A. $D_A(1)$ is the subset of A' given by $D_A(1)=\{f\in A': \|f\|=1=f(1)\}$. For $x\in A$, $\mathrm{Sp}_A(x)$ denotes the spectrum of x in A.

Analogous notations will be used to denote the corresponding sets associated with B.

LEMMA 1. *Let v be an extreme point of B_1. Then v^*v is an idempotent.*

PROOF. The proof is contained in [2, Theorem 1]. It is shown that, if C is the closed subalgebra in B generated by 1 and v^*v, then v^*v, regarded as a function on the carrier space of C, can assume no values different from 0 and 1. The result follows immediately.

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LEMMA 2. *Let u be a unitary element of A. Then Tu is neither a left nor a right divisor of zero in B.*

PROOF. Let u be a unitary element of A and let $x \in B$ for which $(Tu)x = 0$. Since T maps A onto B, $\exists y \in A$ such that $x = (Ty)^*$. Hence $(Tu)(Ty)^* = 0 = (Ty)(Tu)^*$. Let $\alpha \in \boldsymbol{C}$. Then

$$\begin{aligned} \|u + \alpha y\|^2 &= \|Tu + \alpha Ty\|^2 = \|(Tu + \alpha Ty)((Tu)^* + \bar{\alpha}(Ty)^*)\| \\ &= \|(Tu)(Tu)^* + |\alpha|^2(Ty)(Ty)^*\| \leqq \|Tu\|^2 + |\alpha|^2\|Ty\|^2. \end{aligned}$$

This gives $\|u+\alpha y\| \leqq (1+|\alpha|^2 k)^{1/2}$, where $k = \|Ty\|^2$. Since u is unitary, $\|1+\alpha u^* y\| = \|u+\alpha y\| \leqq (1+|\alpha|^2 k)^{1/2}$. It follows that as $\alpha \to 0$ with $\alpha \in \boldsymbol{R}$, we have both

$$\|1 + \alpha u^* y\| \leqq 1 + o(\alpha), \qquad \|1 + i\alpha u^* y\| \leqq 1 + o(\alpha).$$

Therefore $u^* y \in H(A) \cap iH(A) = (0)$.

Since u^* is regular, $y = 0$. Thus $x = (Ty)^* = 0$. Hence, Tu is not a left divisor of zero in B, and it may be similarly shown that Tu is not a right divisor of zero in B.

It is obvious that Lemma 2 remains true if Tu is replaced by $(Tu)^*$ in its statement.

LEMMA 3. *Let u be a unitary element of A. Then Tu is a unitary element of B.*

PROOF. Since T is a linear isometry of A onto B, T maps the extreme points of A_1 onto the extreme points of B_1. By [**1**, Theorem 3], u is a vertex, and hence an extreme point, of A_1. Thus Tu is an extreme point of B_1.

Let $p = (Tu)^* Tu$. It follows easily from Lemma 2 that p is not a divisor of zero.

Now, since by Lemma 1, p is an idempotent in B, we have $p(p-1) = 0$. Hence $p = 1$, i.e. $(Tu)^* Tu = 1$.

Now, if $y \in B$ is an extreme point of B_1, y^* is also an extreme point of B_1. Hence $(Tu)^*$ is an extreme point of B_1. Applying the above argument to $(Tu)^*$ instead of to Tu, it is clear that $(Tu)(Tu)^* = 1$. Hence Tu is a unitary point of B.

THEOREM. *Let A and B be B^*-algebras each containing an identity* 1. *Then if T is a linear isometry mapping A onto B, there exists a unitary element v of B and a C^*-isomorphism τ of A onto B such that $T = v\tau$.*

PROOF. Let T be a linear isometry of A onto B. By Lemma 3, $T1$ is a unitary element of B. Let $v = T(1)$, and define the mapping τ of A into B by $\tau = v^* T$. τ is clearly linear, and maps A onto B. Further, since v^* is unitary, $\|\tau(x)\| = \|v^*T(x)\| = \|Tx\| = \|x\|$ for x in A. Thus τ is an isometry, and $\tau(1) = v^*v = 1$.

Since $T = v\tau$, if we can show that τ is a C^*-isomorphism, the theorem will be proved. Let $h \in H(A)$. Then

$$\|1 + i\alpha\tau(h)\| = \|1 + i\alpha h\| \leqq 1 + o(\alpha) \qquad (\alpha \in \boldsymbol{R}, \alpha \to 0).$$

Thus $\tau(h) \in H(B)$ and τ is a *-mapping.

Finally, we must prove that $\tau(x^2) = [\tau(x)]^2$ $(x \in A)$. Let $h \in H(A)$, $\alpha \in \boldsymbol{R}$. Then $e^{i\alpha h}$ is a unitary element of A. By Lemma 3 applied to τ, $\tau(e^{i\alpha h})$ is a unitary element of B, i.e. $\tau(e^{i\alpha h})\tau(e^{-i\alpha h}) = 1$.

Thus $[1 + i\alpha\tau(h) - \alpha^2\tau(h^2)/2][1 - i\alpha\tau(h) - \alpha^2\tau(h^2)/2] = 1 + O(\alpha^3)$ as $\alpha \to 0$, using the fact that τ is continuous and $\tau(1) = 1$.

Hence $1 + \alpha^2[(\tau(h))^2 - \tau(h^2)] = 1 + O(\alpha^3)$ as $\alpha \to 0$; i.e. $[\tau(h)]^2 - \tau(h^2) = O(\alpha)$ as $\alpha \to 0$; $[\tau(h)]^2 = \tau(h^2)$.

Now let $x \in A$, $x = h + ik$, where $h, k \in H(A)$. Since $[\tau(h+k)]^2 = \tau[(h+k)^2]$, we have

$$\tau(hk + kh) = \tau(h)\tau(k) + \tau(k)\tau(h).$$

Hence $\tau(x^2) = \tau(h^2 - k^2 + i(hk + kh)) = [\tau(x)]^2$ $(x \in A)$. Thus τ is a C^*-isomorphism.

NOTE. The converse of the above theorem was also proved by Kadison [2], i.e. if τ is a C^*-isomorphism mapping A onto B and v is a unitary element in B, then $T = v\tau$ is a linear isometry of A onto B. This result is an easy consequence of [**4**, Corollary 1].

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