THE SPECTRUM OF A LINEAR OPERATOR UNDER PERTURBATION BY CERTAIN COMPACT OPERATORS

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Let T be a bounded linear operator on a Banach space X. A subset of the spectrum of T which is invariant under certain compact perturbation of T is studied. It consists of the spectrum of T with finite-dimensional poles deleted. In the case of a bounded operator, it coincides with the essential spectrum as defined by F. E. Browder [1]. It is characterized as a set considered by Caradus [2]. A formula of the spectral radius type is proved. Furthermore, a spectral mapping theorem is valid.

The notation is that of Taylor [5]. Let $R(T)$ denote the range of T and $N(T)$ the nullspace of T, i.e., $N(T)=\{x:Tx=0\}$. The dimension of $N(T)$, $n(T)$, is called the nullity of T and the codimension of $R(T)$, $d(T)$, the defect of T. Suppose for some integer k, $N(T^k)=N(T^{k+1})$; then the ascent, $\alpha(T)$, is defined as the smallest value of k for which this is true. The smallest integer for which $R(T^K)=R(T^{K+1})$ is called the descent of T and is denoted by $\delta(T)$. For the operator $\lambda-T$, $n(\lambda-T)$ is abbreviated to $n(\lambda)$, etc. $B(X)$ will denote the bounded linear operators, $C(X)$ the compact linear operators. $A\perp B$ means $AB=BA=0$. Let $[T]\in B(X)/C(X)$; then $\sigma([T])$ denotes the spectrum of $[T]$ as an element of that Banach algebra. For a linear operator T, let $P(T)=\{C\in C(X):T-C\perp C\}$ and $Q(T)=\{D\in C(X): DT=TD\}$. The object of this paper is to study the sets

$$\sigma_{P(T)} = \bigcap_{C\in P(T)} \sigma(T-C), \qquad \sigma_{Q(T)} = \bigcap_{D\in Q(T)} \sigma(T-D).$$

The complement of $\sigma_{P(T)}$ will be denoted by $\rho_{P(T)}$, and the complement of $\sigma_{Q(T)}$ will be denoted by $\rho_{Q(T)}$. When no confusion will arise, the T will be suppressed.

LEMMA 1. *σ_P is a closed set, and $\sigma([T])\subseteq\sigma_P\subseteq\sigma(T)$.*

PROOF. σ_P is closed because it is the intersection of closed sets. Since $0\in P$, then $\sigma_P\subseteq\sigma(T-0)=\sigma(T)$.

Let $\lambda\in\rho_P$; then there is a $C\in P$ such that $\lambda\in\rho(T-C)$. Thus, $R_\lambda(\lambda-T+C)=I$, where $R_\lambda=(\lambda-T+C)^{-1}$, the resolvent operator. Then $[R_\lambda][\lambda-T]=[\lambda-T][R_\lambda]=[I]$. This implies that $\lambda\in\rho([T])$, and $\rho_P\subseteq P([T])$. Hence, $\sigma([T])\subseteq\sigma_P$.

Received by the editors March 28, 1967 and, in revised form, November 14, 1968.

LEMMA 2. *Let $T \in B(X)$. Suppose $\lambda_0 \neq 0$ is an isolated point of $\sigma(T)$. Let E_0 be the spectral projection associated with λ_0. Then $T - TE_0 \perp TE_0$* and $\lambda_0 \notin \sigma(T - TE_0)$.

PROOF. The operational calculus for T (see [4]) implies that $TE_0 = E_0 T$ and $(I - E_0)E_0 = E_0(I - E_0) = 0$. These statements give $T - TE_0 \perp TE_0$.

Let $f(\lambda) = \lambda$ on a neighborhood of $\sigma(T) \sim \{\lambda_0\}$ and $f(\lambda) = 0$ on a neighborhood of $\{\lambda_0\}$. Then $f \in \mathfrak{A}_\infty(T)$, and $f(T) = T - TE_0$. The spectral mapping theorem implies $\lambda_0 \notin \sigma(T - TE_0)$.

THEOREM 1. (a) $\rho_{P(T)} \sim \{0\} = \{\lambda : n(\lambda) = d(\lambda)$ *and* $\delta(\lambda) = \alpha(\lambda)\} \sim \{0\}$;

(b) $\rho_{Q(T)} = \{\lambda : n(\lambda) = d(\lambda)$ *and* $\delta(\lambda) = \alpha(\lambda)\}$.

PROOF. Let $\lambda \in \rho_Q$; then there is a $D \in Q$ such that $\lambda \in \rho(T - D)$. We can write $\lambda - T = (\lambda - (T - D)) + (-D)$. Let $U = \lambda - (T - D)$. Then U has the properties that it has a bounded inverse, $(\lambda - T) - U$ is compact, and $(\lambda - T)U = U(\lambda - T)$ (since $TD = DT$). Thus, Theorem 6.3 of Yood [6] implies that $n(\lambda) = d(\lambda)$ and $\alpha(\lambda) = \delta(\lambda)$. Also, $\rho_P \subset \rho_Q$.

Let $\lambda \in \sigma(T)$ such that $n(\lambda) = d(\lambda)$ and $\alpha(\lambda) = \delta(\lambda)$. Now Theorem 9.4 of Taylor [5] shows that λ is an isolated point of $\sigma(T)$. Then by Corollary 9.3 of Taylor [5], we conclude that E_λ, the associated spectral projection, is a finite-dimensional operator. Thus, TE_λ is compact. If $\lambda \neq 0$, then Lemma 2 implies $T - TE_\lambda \perp TE_\lambda$ and $\lambda \notin \sigma(T - TE_\lambda)$. H nce $\lambda \in \rho_P \sim \{0\}$. Thus we have proved (a).

To prove (b), it suffices from the above to consider $\lambda = 0$. For $\mu \neq 0$, $T_\mu = \mu - T$ has a finite-dimensional pole at μ, and the associated spectral projection $E_\mu = E_0$, by Theorem 5.71D of Taylor [4]. By Lemma 2, $\mu \notin \sigma(T_\mu - T_\mu E_0)$. Hence $(\mu - (T_\mu - T_\mu E_0))^{-1} = (T + T_\mu E_0)^{-1}$ exists, and $-T_\mu E_0 \in Q$. This proves (b).

Caradus [2] defined the Riesz region, $\mathfrak{R}_T$, of T to be $\{\lambda : \alpha(\lambda)$ and $\delta(\lambda)$ are finite$\}$; the Fredholm region, $\mathfrak{F}_T$, to be $\{\lambda : n(\lambda)$ and $d(\lambda)$ are finite$\}$.

COROLLARY 1. $\rho_{Q(T)} = \mathfrak{R}_T \cap \mathfrak{F}_T$. *Hence $\mathfrak{R}_T \cap \mathfrak{F}_T$ is open.*

PROOF. Theorem 6.1 of Yood [6] or Lemma 2 of Caradus [2] imply that

$$\mathfrak{R}_T \cap \mathfrak{F}_T = \{n(\lambda) = d(\lambda) \text{ and } \alpha(\lambda) = \delta(\lambda)\}.$$

Theorem 1 completes the proof.

COROLLARY 2. *$\lambda \in \sigma_Q(T)$ if and only if either λ is a limit point of $\sigma(T)$, or λ is an isolated point whose associated spectral projection is infinite dimensional.*

PROOF. Theorem 1, and Theorem 9.3 and Corollary 9.3 of Taylor [5], imply that the points of $\rho_Q \cap \sigma(T)$ are isolated points whose spectral projections are finite-dimensional operators.

Let $r = \sup |\lambda|$ for $\lambda \in \sigma_P(T)$. Then the following spectral radius type theorem is valid.

THEOREM 2.

$$r = \lim_n \{ \inf_{C \in P} \|T^n - C^n\| \}^{1/n}.$$

PROOF. Since $T - C \perp C$, we have by induction $(T-C)^n = T^n - C^n$. Let $r(A)$ be the spectral radius of $A \in B(X)$. It is well known that $r(A^n) = (r(A))^n$ and $\|A^n\| \geqq (r(A))^n$. Hence, for $C \in P$

$$\|(T - C)^n\| \geqq (r(T - C))^n \geqq r^n.$$

For each n,

$$\{ \inf_{C \in P} \|T^n - C^n\| \}^{1/n} \geqq r.$$

Let $a > r$. Pick p such that $a > p > r$. Then if $|\lambda| > p$, we have $n(\lambda) = d(\lambda)$ and $\alpha(\lambda) = \delta(\lambda)$. If $\lambda \in \sigma(T)$ and $|\lambda| > p$, then Theorem 9.4 of Taylor [5] implies that λ is an isolated point of $\sigma(T)$, and Corollary 9.3 of Taylor [5] that the associated spectral projection is a finite dimensional operator.

There can only be a finite number of such points $\lambda \in \sigma(T)$ and $|\lambda| > p$ (for Theorem 9.4 of Taylor [5] would imply that a limit point of such points would be isolated). Denote these points by $\{\lambda_i\}_1^n$. Let E_i be the finite-dimensional projection associated with λ_i. Then the operational calculus for T gives $C = T(\sum_1^n E_i) \in P$, and the spectral mapping theorem that $\lambda_i \notin \sigma(T - C)$ for $i = 1, \cdots, n$. Hence, $p \geqq r(T - C)$.

Thus, by the spectral radius theorem there is an N such that $a > \|(T-C)^n\|^{1/n} \geqq r$ for $n \geqq N$. Thus $a^n > \|(T-C)^n\| \geqq r^n$. But $\|(T-C)^n\| \geqq \|T^n - C^n\| \geqq \inf_{C \in P} \|T^n - C^n\|$. Hence,

$$a^n > \inf_{C \in P} \|T^n - C^n\| \geqq r^n, \quad \text{or} \quad a > \{ \inf_{C \in P} \|T^n - C^n\| \}^{1/n} \geqq r,$$

which completes the proof.

The norm in the Banach algebra $B(X)/C(X)$ is given by $K(T) = \inf_C \|T - C\|$ where $C \in C(X)$. The next theorem shows the spectral radius of an element of $B(X)/C(X)$ is r.

THEOREM 3. *For any* $T \in B(X)$,

$$r = \lim_{n\to\infty} [K(T^n)]^{1/n}.$$

PROOF. Let $s = \lim_{n\to\infty} [K(T^n)]^{1/n}$. Then s is the spectral radius of the element $[T]$ in $B(X)/C(X)$. Since $G = \{\lambda : |\lambda| > s\}$ is an open connected set, Theorem 3.3 and its corollary of Gohberg and Krein [**3**] imply that $\sigma(T) \cap G$ consists of isolated points of $\sigma(T)$ such that $n(\lambda) < \infty$. Hence, Corollary 9.3 of Taylor [**5**] implies that the spectral projections associated with each of these is finite dimensional. Let l be arbitrary and $l > s$. Then there are only a finite number of points $\lambda \in \sigma(T)$ and $|\lambda| \geqq l$. Let σ denote the spectral set consisting of these points. Let E_σ be the spectral projection associated with σ. Then, as before, $T - TE_\sigma$ has spectrum inside the circle $|\lambda| = l$. TE_σ is a finite dimensional operator. Thus $l > r$. Lemma 1 implies that $r \geqq s$. Hence $r = s$.

The operational calculus of an operator T allows one to assign an operator $f(T)$ for every function f analytic on a neighborhood of $\sigma(T)$ (see Taylor [**4**]). The following type of "spectral mapping" theorem is valid.

THEOREM 4. *Let f be analytic on an open set containing $\sigma(T)$. Suppose for each λ_0 that $\{\lambda : f(\lambda) = f(\lambda_0)\}$ is finite. Then $f(\sigma_Q(T)) = \sigma_Q(f(T))$.*

PROOF. Suppose $\lambda_0 \in \sigma_Q(T)$. Since the spectral mapping theorem implies that $f(\sigma(T)) = \sigma(f(T))$, $f(\lambda_0)$ is either a limit point of $\sigma(f(T))$ or an isolated point. If $f(\lambda_0)$ is a limit point, Corollary 2 implies that $f(\lambda_0) \in \sigma_Q(f(T))$. If $f(\lambda_0)$ is isolated, then Theorem 5.71D of Taylor [**4**] implies that $\sigma = \{\lambda | f(\lambda) = f(\lambda_0)\} \cap \sigma(T)$ is a finite spectral set of T, and the spectral projection associated with σ and T, $E_\sigma(T)$, equals that associated with $f(\lambda_0)$ and $F_{f(\lambda_0)}(f(T))$, i.e. $E_\sigma(T) = F_{f(\lambda_0)}(f(T))$. Since σ is a finite spectral set, this implies that λ_0 is an isolated point. Corollary 2 implies that E_{λ_0} is infinite dimensional. Hence $F_{f(\lambda_0)}$ is infinite dimensional. Thus $f(\lambda_0) \in \sigma_Q(f(T))$, or $f(\sigma_Q(T)) \subset \sigma_Q(f(T))$.

Suppose that $\mu \in \sigma_Q(f(T))$. If μ is a limit point of $\sigma(f(T))$, then since $f(\sigma(T)) = \sigma(f(T))$, there is a limit point λ of $\sigma(T)$ such that $f(\lambda) = \mu$. Corollary 2 implies that $\lambda \in \sigma_Q(T)$. If μ is isolated, then, as before, $\sigma = \{\lambda | f(\lambda) = \mu\} \cap \sigma(T)$ is a nonempty finite spectral set such that $E_\sigma(T) = F_\mu(f(T))$. Since points of σ are isolated, E_σ is the finite sum of the spectral projections associated with the points of σ. Since F_μ is infinite dimensional, one of these projections must be infinite dimensional. Thus there is a $\lambda \in \sigma$ such that $f(\lambda) = \mu$ and $\lambda \in \sigma_Q(T)$. Thus, $f(\sigma_Q(T)) = \sigma_Q(f(T))$.

REMARK. The above theorems hold if $P(T)$ and $Q(T)$ are replaced with finite-dimensional operators that satisfy the defining conditions for these sets.

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