COMPLETELY CONTINUOUS INVERSES OF ORDINARY DIFFERENTIAL OPERATORS

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Many important properties, such as the absence of essential spectrum, follow when an ordinary differential operator comes from a differential expression such that the minimal operator has completely continuous inverse. This always happens in the case of a compact interval. On infinite intervals it happens much less frequently, however. To assure that the minimal operator is 1-1, we analyze the question only on intervals of type $I=[a, \infty)$.

By an ordinary differential expression τ we mean an expression of type $\sum_0^n a_k D^k$, where $a_k \in C^k(I)$, and a_n is nonvanishing on I. With $1<p<\infty$ and $1<q<\infty$, we define the minimal operator $T_{0,p,q}$ in the usual fashion. (See Goldberg's book for details.)

Our main result shows roughly that for formally selfadjoint differential expressions, $T_{0,2,2}$ has compact inverse if and only if the spectra of its selfadjoint extensions are completely dependent on the boundary conditions. A weaker result holds for $T_{0,p,p}$ with $p\neq 2$.

THEOREM 1. *Let τ be a formally selfadjoint differential expression on I. Then the essential spectrum of the minimal operator $T_{0,2,2}$ is null if and only if $T_{0,2,2}^{-1}$ is completely continuous from range $T_{0,2,2}$ into $L_2(I)$.*

PROOF. Suppose $T_{0,2,2}$ has no essential spectrum. Pick a selfadjoint extension T of $T_{0,2,2}$ such that T is 1-1. T will have no essential spectrum. Then $T=\sum_1^\infty \lambda_i P_i$ by the spectral theorem, where each P_i is a projection on a finite-dimensional subspace and the λ_i do not cluster. Thus, $T^{-1}=\sum_1^\infty (1/\lambda_i)P_i$ and $\{1/\lambda_i\}$ has no cluster point except 0. Clearly, T^{-1} is completely continuous and therefore $[T_{0,2,2}]^{-1}$ is. The converse is clear. This completes the proof.

THEOREM 2. *Let τ be a differential expression with formal adjoint τ^+. Let $1/p+1/q=1$. If every solution of $\tau f=0$ is in $Lp(I)$, and every solution of $\tau^+ f=0$ is in Lq (I), then the minimal operator $T_{0,p,p}$ has completely continuous inverse.*

PROOF. $T_{0,p,p}^{-1}f(x)=\int_a^x K(x,y)f(y)dy$, where $K(x,y)=\sum_1^n f_i(x)\bar{g}_i(y)$, with $\tau f_i=0$ and $\tau^+ g_i=0$. Complete continuity follows from this and the proof is completed.

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THEOREM 3. *If τ has uniformly bounded coefficients then its minimal operator $T_{0,p,p}$ does not have compact inverse.*

PROOF. Suppose f is infinitely differentiable and is of compact support in (a, ∞). Then each derivative of f is in $Lp[a, \infty)$. Further, $\|\tau f\| \leqq \sum_0^n M_i \|f^i\|p$ where T is of nth order and M_i is a fixed positive member for each i. This inequality also holds for all translates of f. Pick any sequence f_j of translates of f which has no convergent subsequence in $Lp(I)$. The set of τf_j is bounded in p norm and thus the inverse of $T_{0,p,p}$ is not completely continuous. This completes the proof.

COROLLARY 1. *If τ has uniformly bounded coefficients, it can not be true that every solution of $\tau f = 0$ is in Lp and every solution of $\tau^+ f = 0$ is in Lq, where $1/p + 1/q = 1$.*

COROLLARY 2. *If τ is classically selfadjoint and has uniformly bounded coefficients, every selfadjoint extension of $T_{0,2,2}$ has nonnull essential spectrum.*

The following theorem will be used to show that we can put any real number into the spectrum of some extension $T_{0,p,p}$ if τ is classically selfadjoint and the minimal operator $T_{0,p,p}$ has completely continuous inverse.

THEOREM 4. *Suppose τ generates a minimal operator $T_{0,p,q}$. Suppose $T_{0,p,q}$ has closed range, and τ is of order n. Then the sum of the number of Lp solutions of $\tau f = 0$ and the number of L_t solutions of $\tau^+ f = 0$ is $\geqq n$, where $1/q + 1/t = 1$.*

PROOF. Let r be the number of non L_t solutions of $\tau^+ f = 0$. Let $\{f_i\}_1^r$ be chosen as follows: f_i is of compact support in (a, ∞), and $\int_I f_i(x)\bar{g}_j(x)dx = a_{ij}$, where $a_{ij} = 0$ if $i \neq j$, $a_{ii} \neq 0$ for each i, and $\{g_j\}_1^r$ are the non L_t solutions of $\tau^+ f = 0$. Further assume $\int_I f_i(x)\bar{\psi}(x)dx$ is 0 for all L_t solutions ψ of $\tau^+\psi = 0$. Now since range $T_{0,p,q}$ is closed, we know that the minimal operator $T_{0,p,q}$ is onto the orthogonal complement of the Lq solution space of τ^+, and thus there are functions h_i in $D(T_{0,p,q})$ with $\tau(h_i) = f_i$. Since f_i is of compact support, h_i is an Lp solution of τ on an interval of form $[c, \infty)$, where $c =$ lub support f_i. Thus h_i can be extended to an Lp solution d_i of τ on all of I. Suppose the d_i are linearly dependent. Then $\sum_1^r \lambda_i d_i = 0$. $\sum \lambda_i h_i = \sum \lambda_i (h_i - d_i)$. $\int_I [\tau(\sum \lambda_i h_i)](x)\bar{g}_j(x)dx = \lambda_j a_{jj}$. Since the λ_j can not all be 0, we have a compact support function $\sum \lambda_i(h_i - d_i) = \sum \lambda_i h_i$ in $D(T_{0,p,q})$ with $\int_I \tau(\sum \lambda_i h_i)(x)\bar{g}_j(x)dx \neq 0$. This contradicts Green's formula, since the first $n-1$ derivatives of h_i vanish at a for each i.

Therefore the number of Lp solutions of $\tau f=0$ plus the number of Lt solutions of $\tau^+ f=0$ is at least the order of τ, by the above theorem, where $1/t+1/q=1$. This completes the proof.

THEOREM 5. *If τ is formally selfadjoint, and the minimal operator as considered from Lp to Lp has closed range, then there are nontrivial Lp solutions of $\tau f=0$.*

PROOF. Under the hypotheses we see that the number of Lp solutions of τ and the number of Lq solutions of τ add up to a number which is at least n, the order of τ. Here $1/q+1/p=1$. If there were not any Lp solutions, then all solutions would be Lq. Thus, the inverse of the minimal operator from Lq to Lp is given by $(T_{0,q,p})^{-1}f(x) = \int_a^x h_i(x)\bar{h}_j(y)f(y)dy$ where all h_i are in Lq. We then see that $(T_{0,q,p})^{-1}$ is continuous and thus range $T_{0,q,p}$ is closed.

But therefore range $T_{0,p,q}$ is closed. Thus the nonexistence of Lp solutions contradicts Theorem 4. This completes the proof.

Now we proceed to our characterizations of minimal operators with completely continuous inverse, assuming τ is formally selfadjoint. In the L_2-L_2 case, by Theorem 1 the property occurs if and only if the minimal operator has no essential spectrum. Thus, if the property holds for any real number λ, there is a 1-1 selfadjoint extension T of $T_{0,2,2}-\lambda I$. Since $T_{0,2,2}$ has no essential spectrum, $\lambda\notin$spectrum T. Now by Theorem 5 and Lemma 9, p. 1398, Dunford and Schwartz, each of the equal deficiency indices of $(T_{0,2,2}-\lambda I)$ is at least 1, and thus by Theorem 10, p. 1400 of the same book, there are selfadjoint extensions which include λ in the spectrum if the property holds. Now if the property does not hold, then $T_{0,2,2}$ has some essential spectrum, so any selfadjoint extension also has this essential spectrum.

In the case $p\neq 2$, if the property holds there is no essential spectrum and consequently there are (see Theorem VI. 4.4, Goldberg, p. 150) 1-1 surjective extensions of $T_{0,p,p}-\lambda I$ for every λ, and λ is not in the spectrum of any of these. By Theorem 5 there are also extensions which include any real λ in the spectrum. It is an interesting question whether the lack of essential spectrum implies complete continuity of the inverse of the minimal operator $T_{0,p,p}$, with $p\neq 2$.

REFERENCES

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