TAMING A SURFACE BY PIERCING WITH DISKS

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In [3] and [2], respectively, Bing proves the following two theorems.

THEOREM 1. *A 2-sphere S in E^3 is tame from complementary domain U if and only if U is* 1-ULC.

THEOREM 2. *If S is a 2-sphere in E^3 and U is a complementary domain of S then there exists a* 0*-dimensional F_σ set $F \subset S$ such that $U \cup F$ is* 1-ULC. *Furthermore if $\{X_i\}$ is a sequence of sets in S, each of which is either a tame finite graph or a tame Sierpinski curve, then F may be chosen in $S - \bigcup X_i$.*

The above two theorems suggest a procedure for showing that a given condition restricting the embedding of S implies S is tame from U. Namely, it may be possible to use the condition to slightly adjust a map f from a disk D into $U \cup F$ so that the new image of D lies entirely in U while $f | \text{Bd } D$ is unaltered. The facts that $f(D) \cap F$ is compact 0-dimensional and F lies in $S - \bigcup X_i$ may also be helpful while adjusting f.

The above technique is employed in this paper to answer in the affirmative the following question asked in [1] and [5]. Is a 2-sphere in E^3 tame if it can be pierced at each arc with a tame disk? Other illustrations of this procedure may be found in [7] and [8].

DEFINITION. A disk D is said to pierce sphere S at arc $A \subset S$ if Bd $A \subset$ Bd D, Int $A \subset$ Int D and the two components of $D - A$ lie in different complementary domains of S.

DEFINITION. If J is a simple closed curve in 2-sphere S and U is a complementary domain of S then J can be collared from U by A if A is an annulus such that $J \subset$ Bd A and $A - J \subset U$.

The reader is referred to [2] and [3] for definitions of other terms used in this paper.

THEOREM 3. *If S is a 2-sphere in E^3 then S is tame if and only if each simple closed curve in S can be collared from each complementary domain of S by a tame annulus.*

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PROOF. Let U be a complementary domain of S. We will show that U is 1-ULC and apply Theorem 1 to conclude that S is tame from U. Suppose $\epsilon > 0$, then it follows from Theorem 2 that there exists a $\delta > 0$ such that if f is a map of the boundary of a disk D into a δ-subset in U then f may be extended to D so that

(1) $f(D)$ is an $\epsilon/2$ subset of $\overline{U}$,

(2) $f(D) \cap S$ is 0-dimensional, and

(3) $f(D) \cap S \subset \operatorname{Int} A$ where A is an $\epsilon/2$-annulus in S.

It follows from (2) that there exists a homeomorphism h of $\{(x, y) \mid 1 \leqq x^2 + y^2 \leqq 4\}$ onto A such that the arc

$$B = h(\{(x, y) \mid 1 \leqq x \leqq 2, y = 0\})$$

misses $f(D)$. From the hypothesis there exists a disk C that pierces S at B and also misses $f(D)$. It follows from (1) and (3) that there exists an open set N such that $N \cap S = \operatorname{Int} A$, $N \cap (f(\operatorname{Bd} D) \cup \operatorname{Bd} C) = \phi$, and Diam $f(D) \cup N < \epsilon$. The proof will be completed by adjusting the singular disk $f(D)$ in N so that the new image of D lies in U.

It follows from the hypothesis that for each $t \in (1, 2)$ there exists a tame annulus A_t such that

(4) $A_t \cap S$ is the simple closed curve $J_t = h(\{(x, y) \mid x^2 + y^2 = t^2\})$, and

(5) $A_t \subset N \cap \overline{U}$.

We need the following well-known lemma from general topology. An easy proof may be obtained by showing that the space of all continuous functions from an element of $\mathcal{C}$ into X is a separable metric space under the sup-norm metric. The elements of $\mathcal{C}$ may then be considered as points in this separable metric space.

LEMMA 4. *If $\mathcal{C}$ is an uncountable collection of homeomorphic compact subsets of a separable metric space X then there exists a countable subcollection $\mathcal{C}'$ of $\mathcal{C}$ such that if $Y \in \mathcal{C} - \mathcal{C}'$ then there exists a sequence $\{Y_i\} \subset \mathcal{C} - \mathcal{C}'$ which converges homeomorphically to Y.*

Apply Lemma 4 by letting $\mathcal{C} = \{A_t \mid t \in (1, 2)\}$ and consider an $A_t \in \mathcal{C} - \mathcal{C}'$. There is a sequence $\{A_{t_i}\} \subset \mathcal{C} - \mathcal{C}'$ converging homeomorphically to A_t; consequently, there exists an integer i such that A_t and A_{t_i} are homeomorphically so close that there exists a singular annulus B_t (resulting from a homotopy) such that

(6) $\operatorname{Bd} B_t = (\operatorname{Bd} A_t \cup \operatorname{Bd} A_{t_i}) - S$, and

(7) $B_t \subset N \cap U$.

The union of A_t, B_t and A_{t_i} is a singular annulus which has no singularities near its boundary. Dehn's lemma [9] is applied to replace $A_t \cup B_t \cup A_{t_i}$ with a nonsingular annulus C_t with the same boundary.

Hence there exists a countable set $P\subset(1, 2)$ such that if $t\in(1, 2)-P$ there exists a tame annulus C_t such that

(8) Bd $C_t=J_t\cup J_s$ for some $s\in(1, 2)$, and

(9) Int $C_t\subset N\cap U$.

Let D_t be the annulus lying in A with the same boundary as C_t. It is straightforward to show that there exists a countable set $Q\subset(1, 2)$ such that $s\in Q$ whenever

$$J_s \not\subset \bigcup_{t\in(1,2)-P} \operatorname{Int} D_t.$$

Furthermore there exists a countable set $R\subset(1, 2)-P$ such that

$$\bigcup_{t\in(1,2)-P} \operatorname{Int} D_t = \bigcup_{t\in R} \operatorname{Int} D_t.$$

The set

$$Y = \left(\bigcup_{s\in Q} J_s\right) \cup \left(\bigcup_{t\in R} \operatorname{Bd} D_t\right)$$

is the union of a countable number of tame simple closed curves, so by Theorem 2 there exists a 0-dimensional set $F\subset S-Y$ such that $U\cup F$ is 1-ULC.

It follows that there exists an open set V containing $f(D)\cap S$ such that loops in $V\cap U$ can be shrunk to points in $(U\cup F)\cap(N-C)$. It is straightforward to find a finite collection $E_1, \cdots, E_k$ of disjoint disks in D such that $f^{-1}(S)\subset\bigcup_{i=1}^k E_i\subset\operatorname{Int} D$ and $f(\operatorname{Bd} E_i)\subset V\cap U$. The map $f|\operatorname{Bd} E_i$ is extended to a map $f_i\colon E_i\to(U\cup F)\cap(N-C)$. The maps f_i $(i=1, \cdots, k)$ and $f|D-\bigcup_{i=1}^k E_i$ are pieced together to form a map $g\colon D\to U\cup F$. Note that $g(D)\cap S\subset F\cap\operatorname{Int} A\subset(\bigcup_{t\in R}\operatorname{Int} D_t)-Y$.

There exists a finite set $T\subset R$ such that $g(D)\cap S\subset\operatorname{Int}_{t\in T} D_t$. We assume that C_t and $C_{t'}$ are in general position whenever $t, t'\in R$ and $t\neq t'$. Note that for each $s\in T$ a simple closed curve K in Int C_s links Bd C if and only if K separates the boundary components of C_s in C_s; thus, a simple closed curve in Int $C_t\cap$Int $C_{t'}$ bounds a disk in Int C_t if and only if it bounds a disk in Int $C_{t'}$. Using standard disk and annulus trading techniques we alter the collection $\{C_t\}_{t\in T}$ to form a new finite collection $\{F_i\}_{i=1}^n$ of annuli such that

(10) Bd $F_i=J_s\cup J_t$ for some $s, t\in(1, 2)$,

(11) Int $F_i\subset N\cap U$,

(12) Int $F_i\cap$Int $F_j=\phi$ whenever $i\neq j$, and if G_i is the annulus lying in A with the same boundary as F_i then

(13) $g(D)\cap S\subset\bigcup_{i=1}^n \operatorname{Int} G_i$.

For $i=1, \cdots, n$ each component of the boundary of F_i links Bd C and $F_i\cap$Bd $C=\phi$; consequently, each simple closed curve in (Int F_i)

$-C$ bounds a disk in Int F_i. Since $g(D)\cap C=\phi$, it follows that there exists a finite collection D_{ij} of disjoint disks in Int F_i such that $g(D)\cap F_i\subset \bigcup_j D_{ij}$. The union, $\bigcup_{i,j} D_{ij}$, separates $g(\mathrm{Bd}\ D)$ from $g(D)\cap S$ on $g(D)$ so it follows from the Tietze Extension Theorem as indicated in [**4**, Lemma 6] that there is a map $f'\colon D\to(U\cap N)\cup(f(D)-N)$ such that $f'|\mathrm{Bd}\ D=g|\mathrm{Bd}\ D=f|\mathrm{Bd}\ D$. Since Diam $f(D)\cup N<\epsilon$, Diam $f'(D)<\epsilon$ and we have U is 1-ULC. An application of Theorem 1 completes the proof.

THEOREM 5. *A 2-sphere S in E^3 is tame if and only if S can be pierced at each arc with a tame disk.*

PROOF. The lemma after Theorem 4 of [**6**] shows that the hypothesis of Theorem 3 is satisfied.

R. J. Daverman has recently weakened the hypothesis of Theorem 5 to include the case where S can be pierced at each arc with a singular disk and each arc of S is tame.

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