APPLICATIONS OF ϵ-ENTROPY TO THE COMPUTATION OF n-WIDTHS

J. W. JEROME AND L. L. SCHUMAKER[1]

1. The concepts of ϵ-entropy and n-width of compact sets in Banach spaces play an important role in approximation theory (see [**1**], [**2**], [**3**] and references therein). The entropy and widths of many compact classes of smooth and analytic functions in various well-known function spaces have been computed.

It is known that entropy and n-width are related to each other, e.g., by certain integral inequalities (see [**3**, p. 164]). There is every reason to believe, however, that in general the behavior of one quantity does not determine in a sharp way the behavior of the other.

The purpose of this paper is to show how the implicit relationship between n-width and entropy inherent in certain negative Vituškin type results for nonlinear approximation can be utilized to compute n-widths of some classes of smooth functions. As examples we shall compute the n-widths for the following classes. Let S be an s-dimensional parallelopiped, and let ω be a monotone increasing subadditive function which vanishes at zero. We define $\Lambda^s_{r\omega}$ in $C(S)$ and $\Lambda^{sp}_{r\alpha}$ in $L^p(S)$, $1 \leqq p < \infty$, as

$$\text{(1.1)} \quad \Lambda^s_{r\omega} = \Lambda^s_{r\omega}(M_0, \cdots, M_{r+1}, S) = \{f : f \in C^r(S), \|D^j f\|_\infty \leqq M_j, \\ 0 \leqq j \leqq r, \quad \text{and} \quad \omega(D^r f; t) \leqq M_{r+1}\omega(t)\}$$

$$\text{(1.2)} \quad \Lambda^{sp}_{r\alpha} = \Lambda^{sp}_{r\alpha}(M_0, \cdots, M_{r+1}, S) = \{f : f \in C^r(S), \|D^j f\|_p \leqq M_j, \\ 0 \leqq j \leqq r, \quad \text{and} \quad \omega(D^r f; t) \leqq M_{r+1} t^\alpha\}, \quad 0 < \alpha \leqq 1,$$

where $\omega(D^r f;\, t)$ denotes the modulus of continuity of $D^r f$ and $D^j f$ denotes an arbitrary partial derivative of f of order j.

The n-widths of $\Lambda^s_{r\omega}$ have been computed elsewhere (cf. [**3**]) by other means, but the results for $\Lambda^{sp}_{r\alpha}$ are new.

2. **Vituškin-type results.** In this section we will state, after some initial definitions, theorems due to Vituškin and Lorentz which say in effect that if the n-width of a class is known to be less than ϵ, then n must be at least as large as the ϵ-entropy of that class. These theorems will later be exploited to obtain lower bounds for n-widths.

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By the n-width, $d_n(A)$, of a set A in a Banach space $\mathfrak{X}$ we mean the number

$$d_n(A) = \inf_{\dim \mathfrak{M}=n} \sup_{f\in A} \inf_{g\in\mathfrak{M}} \|f - g\|$$

where $\mathfrak{M}\subset\mathfrak{X}$, and by the ϵ-entropy $H_\epsilon(B)$ in $\mathfrak{X}$ we mean the logarithm of the minimum number of sets of diameter $\leqq 2\epsilon$ whose union contains B. Finally, $\lambda(\epsilon)\approx\mu(\epsilon)$ (weak asymptotic equivalence) is to mean that $\lambda = O(\mu)$ and $\mu = O(\lambda)$ as $\epsilon\to 0$.

We state first the theorem of Vituškin [**1**, Theorem 12, p. 928] which, though in its original form deals with nonlinear approximation, is here specialized to the case of linear approximation of the class $\Lambda^s_{r\omega}$.

THEOREM 2.1 (VITUŠKIN). *Consider the class* $\Lambda^s_{r\omega}\subset C(S)$. *If* $d_n(\Lambda^s_{r\omega}) < \epsilon$ *then*

$$n \geqq c_1 H_\epsilon(\Lambda^s_{r\omega}). \tag{2.1}$$

Lorentz [**1**, Theorem 6, p. 915] has proved analogous results for arbitrary separable Banach spaces:

THEOREM 2.2 (LORENTZ). *Let A be an arbitrary compact set in a separable Banach space $\mathfrak{X}$ and let $d_n(A)<\epsilon$. If $\forall q$, $0<q<1$, $\exists c_1>0$ $\ni H_{c_1\epsilon}(A)\leqq qH_\epsilon(A)$, then*

$$n \geqq c_1 H_{\epsilon 2_\epsilon}(A) - c_2. \tag{2.2}$$

Notice that for sufficiently small ϵ (i.e. sufficiently large n) (2.2) may be rewritten as

$$n \geqq c_3 H_{\epsilon 2_\epsilon}(A). \tag{2.3}$$

3. **n-widths of $\Lambda^s_{r\omega}$.** In this section we illustrate our method by obtaining the following

THEOREM 3.1 (*cf.* [**3**, p. 135]). *The n-width of the class $\Lambda^s_{r\omega}\subset C(S)$ is given by*

$$d_n(\Lambda^s_{r\omega}) \approx n^{-r/s}\omega(n^{-1/s}).$$

PROOF. By the classical Jackson theorem for several variables [**3**, Theorem 8, p. 90] if $n^{1/s}$ is an integer we can find a polynomial P_n of degree $n^{1/s}-1$ in each of its s variables such that

$$\|f - P_n\|_\infty \leqq M_s n^{-r/s}\omega(n^{-1/s})$$

for every $f\in\Lambda^s_{r\omega}$. It follows from this using the subadditivity and monotonicity properties of ω that

$$d_n(\overset{s}{\Lambda}_{r\omega}) = O(n^{-r/s}\omega(n^{-1/s})) \quad \text{as } n \to \infty.$$

To obtain a lower bound for d_n we notice that by Theorem 2.1 if $n < c_1 H_\epsilon(\Lambda^s_{r\omega})$ then $d_n(\Lambda^s_{r\omega}) \geqq \epsilon$. Since (cf. [1, p. 920])

$$\frac{c_2}{\delta(\beta\epsilon)^s} \leqq H_\epsilon(\overset{s}{\Lambda}_{r\omega}) \leqq \frac{c_3}{\delta(\gamma\epsilon)^s}$$

where $\delta = \delta(\eta)$ is defined by the equation $\delta^r\omega(\delta) = \eta$ and c_2, c_3, β, γ are positive constants, then any solution ϵ_n of

$$n = \frac{c_1 c_2}{2\delta(\beta\epsilon_n)^s}$$

provides a lower bound for $d_n(\Lambda^s_{r\omega})$.

This equation may be rewritten as

$$\delta(\beta\epsilon_n) = \left(\frac{n}{c_4}\right)^{-1/s} \qquad (c_4 = c_1c_2/2),$$

and hence recalling that $\delta^r(\beta\epsilon_n)\omega(\delta(\beta\epsilon_n)) = \beta\epsilon_n$ we obtain

$$\beta\epsilon_n = \left(\frac{n}{c_4}\right)^{-r/s}\omega\left(\left(\frac{n}{c_4}\right)^{-1/s}\right).$$

But then

$$d_n(\overset{s}{\Lambda}_{r\omega}) \geqq \epsilon_n \geqq c_5 n^{-r/s}\omega(n^{-1/s})$$

where

$$c_5 = \frac{c_4^{r/s}}{\beta([c_4^{-1/s}] + 1)}.$$

This completes the proof of Theorem 3.1.

4. n-widths of $\Lambda^{sp}_{r\alpha}$.

Theorem 4.1. *The n-width of the class $\Lambda^{sp}_{r\alpha} \subset L^p(S)$ is given by*

$$d_n(\overset{sp}{\Lambda}_{r\alpha}) \approx n^{-(r+\alpha)/s} \qquad (1 \leqq p < \infty).$$

Proof. Since $\|g\|_p \leqq K\|g\|_\infty$ for all $g \in L^p(S)$ we have, by the s-dimensional Jackson theorem,

$$d_n(\overset{sp}{\Lambda}_{r\alpha}) \leqq cn^{-(r+\alpha)/s}$$

for the L^p n-width of $\Lambda^{sp}_{r\omega}$. To compute a lower bound we apply

Theorem 2.2 with $A=\Lambda_{r\omega}^{sp}$ and $\mathfrak{X}=L^p$. The entropy of the class $\Lambda_{r\alpha}^{sp}$ is given by [1, p. 921].

$$H_\epsilon(\Lambda_{r\alpha}^{sp}) \approx \epsilon^{-s/(r+\alpha)}.$$

Since H_ϵ satisfies the hypotheses of Theorem 2.2 the result follows as in §3.

References

1. G. G. Lorentz, *Metric entropy and approximation*, Bull. Amer. Math. Soc. **72** (1966), 903–937.

2. A. G. Vituškin, *Theory of transmission and processing of information*, Pergamon Press, New York, 1961.

3. G. G. Lorentz, *Approximation of functions*, Holt, New York, 1966.

Mathematics Research Center, University of Wisconsin and University of Texas