ON A WEAKLY CONVERGENT SEQUENCE OF NORMAL FUNCTIONALS ON A VON NEUMANN ALGEBRA[1]

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G. F. Dell'Antonio [1] recently has discussed weakly convergent sequences of normal states of von Neumann algebras and proved that every weakly convergent sequence of normal states of a factor of type I converges also uniformly. Moreover, he has shown that this statement is not true for a factor of type II. The purpose of this note is to investigate when a weakly convergent sequence of normal states converges also uniformly in the case of type II factors. We shall confine ourselves to the class of normal generalized irreducible functionals on a factor of type II. Then the generalized irreducibility of functionals makes it possible to find a simple and relevant condition for our problem.

Throughout this paper, for convenience *functional* will always mean a positive linear functional on a von Neumann algebra. Let us recall that a functional ρ on a von Neumann algebra M is said to be *generalized irreducible* on M if whenever ω is a functional on M such that that $\omega \leqq \lambda\rho$ for some positive constant λ (i.e., $\omega(A) \leqq \lambda\rho(A)$ for all positive operators A in M), there exists a positive operator B in M such that $\omega(A) = \rho(AB)$ for all $A \in M$. As is well known, every normal trace of a finite von Neumann algebra is generalized irreducible (see [4, Lemma 14.1]). We say that a sequence $\{\rho_n\}$ of functionals on a von Neumann algebra M is *bounded from below* by a functional ρ on M if $\rho \leqq \rho_n$ for all n. Then we shall prove the following.

THEOREM. *Let $\{\rho_n\}$ be a sequence of normal generalized irreducible functionals on a semifinite factor M bounded from below by a nonzero normal functional ω on M. If ρ_n converges weakly to a normal generalized irreducible functional ρ on M, then ρ_n converges also uniformly to ρ.*

The reader should refer to J. Dixmier's book [2] as a general reference on von Neumann algebras.

1. In what follows, M will denote a semifinite factor on a Hilbert space H with inner product $\langle\ ,\ \rangle$. First we are concerned with a representation theorem of a normal generalized irreducible functional on M which is essentially due to Halpern [3, Proposition 3.1].

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LEMMA 1. *Let ρ be a normal generalized irreducible functional on M. If ρ is faithful on M, then ρ is a trace of M.*

PROOF. Let us consider the faithful representation π of M defined by ρ. Then the representation space is the completion K of the pre-Hilbert space M with inner product $(A, B)=\rho(B^*A)$, and ρ has the form $\rho(A)=(\pi(A)\xi, \xi)$, where ξ is a cyclic vector of $\pi(M)$, i.e., $[\pi(M)\xi]=K$. Moreover, $[\pi(M)'\xi]=K$ since ρ is faithful. Thus we may assume without loss of generality that there exists a vector $\phi\in H$ such that

$$\rho(A) = \langle A\phi, \phi\rangle \quad \text{and} \quad H = [M\phi] = [M'\phi].$$

Then it follows from [3, Proposition 3.1] that ϕ is a trace vector of M, that is, ρ is a trace of M.

LEMMA 2. *Let ρ be a normal functional on M and let τ be a faithful normal trace defined on a two-sided ideal $\mathfrak{M}$ of M containing all of the finite projections in M. Then ρ is generalized irreducible if and only if it is represented in the form*

$$\rho(A) = \tau(\lambda EA) \qquad \textit{for all } A \in M,$$

where E is the support of ρ and λ is a positive constant.

PROOF. The "if" part follows immediately from [4, Lemma 14.1].

For each $A\in M$, we denote by A_E the restriction of EAE to EH and by M_E the restriction of EME to EH. Then ρ induces a faithful normal generalized irreducible functional $\tilde{\rho}$ on M_E by restriction. Indeed, if $\tilde{\omega}$ is a functional on the factor M_E such that $\tilde{\omega}\leqq\mu\tilde{\rho}$ for some positive constant μ, then the functional ω on M defined by $\omega(A)=\tilde{\omega}(A_E)$ $(A\in M)$ is bounded by $\mu\rho$. Thus there is a positive operator B in M such that $\omega(A)=\rho(AB)$ for all $A\in M$, and so

$$\begin{aligned}\tilde{\omega}(A_E) &= \omega(EAE) = \rho(EAEB) = \rho(EAEBE) + \rho(EAEB(I-E)) \\ &= \rho(EAEBE) = \tilde{\rho}(A_EB_E).\end{aligned}$$

Now it turns out from Lemma 1 that $\tilde{\rho}$ is a trace of M_E. Since M_E is a finite factor, there is a positive constant λ such that $\tilde{\rho}=\lambda\tilde{\tau}$, where $\tilde{\tau}$ is the faithful normal trace of M_E induced by the restriction of τ on EME. This means that $\rho(EAE)=\lambda\tau(EAE)$ for all $A\in M$. Therefore we have

$$\rho(A) = \rho(EAE) = \tau(\lambda EA) \qquad \text{for all } A \in M.$$

REMARK. As we have seen above, the support E of a normal generalized irreducible functional on M is necessarily a *finite* projection in M, i.e., $E\in\mathfrak{M}$.

The following lemma is elementary, but is of fundamental importance.

LEMMA 3. *Let* $\{\rho_n\}$ *be a sequence of functionals on a von Neumann algebra. If* $\{\rho_n\}$ *converges weakly to* 0, *then* ρ_n *converges uniformly to* 0.

In fact, $\|\rho_n\| = \rho_n(I) \to 0$ as $n \to \infty$.

2. **Proof of Theorem.** By Lemma 2, ρ_n and ρ are expressed in the forms $\rho_n(A) = \lambda_n \tau(E_n A)$ and $\rho(A) = \lambda \tau(EA)$, where E_n and E are the supports of ρ_n and ρ respectively. Let F be the support of ω. Then F is nonzero and $F \leqq E_n$, E by the hypothesis. Since $\rho_n(F) \to \rho(F)$, $\lambda_n \tau(F) \to \lambda \tau(F)$ and hence $\lambda_n \to \lambda$ as $n \to \infty$.

Now let us consider the factor $\tilde{M} = M_E$ obtained by restricting EME to EH and let $\tilde{A}$ denote an operator in $\tilde{M}$ which is the restriction of EAE to EH. $\tilde{\rho}_n$, $\tilde{\rho}$ and $\tilde{\tau}$ denote the functionals on $\tilde{M}$ induced by the restrictions of ρ_n, ρ and τ on EME respectively. That is, $\tilde{\rho}_n(\tilde{A}) = \rho_n(EAE)$ and $\tilde{\rho}(\tilde{A}) = \rho(EAE)$. Then it is easily seen that $\tilde{\rho}_n(\tilde{A}) = \lambda_n \tilde{\tau}(\tilde{E}_n \tilde{A})$ and $\tilde{\rho}(\tilde{A}) = \lambda \tau(\tilde{E}\tilde{A})$. Here we should notice that $\tilde{E}_n$ are positive operators in $\tilde{M}$ such that $\tilde{E}_n \leqq \tilde{E}$. We shall show that $\tilde{\rho}_n \to \tilde{\rho}$ uniformly as $n \to \infty$. To prove this it is enough to consider the case when $\tilde{M}$ is standard. In this case, as is well known, $\tilde{\tau}$ is expressible as the form $\tilde{\tau}(\tilde{A}) = \langle \tilde{A}\phi, \phi \rangle$, where ϕ is a trace vector of $\tilde{M}$ such that $[\tilde{M}'\phi] = [\tilde{M}\phi] = EH$. Then it follows from the hypothesis that $\langle \lambda_n \tilde{E}_n \tilde{A}\phi, \tilde{A}\phi \rangle \to \langle \lambda \tilde{E}\tilde{A}\phi, \tilde{A}\phi \rangle$ for each $\tilde{A} \in \tilde{M}$. Since $\{\lambda_n \tilde{E}_n\}$ is bounded and $EH = [\tilde{M}\phi]$, the sequence $\{\lambda_n \tilde{E}_n\}$ of positive operators converges weakly to $\lambda \tilde{E}$ in $\tilde{M}$. Thus, having recalled that $\lambda_n \to \lambda$, $\tilde{E}_n \to \tilde{E}$ weakly. This means that the sequence $\{\tilde{E} - \tilde{E}_n\}$ of positive operators converges weakly to 0, and so it converges also strongly to 0. That is to say, $\tilde{E}_n \to \tilde{E}$ strongly. Hence we can conclude that $\lambda_n \tilde{E}_n \to \lambda \tilde{E}$ strongly. Consequently, for each $\epsilon > 0$, there is a positive integer N such that $\|(\lambda_n \tilde{E}_n - \lambda \tilde{E})\phi\| < \epsilon / \|\phi\|$ for all $n \geqq N$. Then

$$\begin{aligned} \|\tilde{\rho}_n - \tilde{\rho}\| &= \sup_{\|\tilde{A}\|=1} |\tilde{\rho}_n(\tilde{A}) - \tilde{\rho}(\tilde{A})| \\ &= \sup_{\|\tilde{A}\|=1} |\langle (\lambda_n \tilde{E}_n - \lambda \tilde{E})\phi, \tilde{A}\phi \rangle| \leqq \|(\lambda_n \tilde{E}_n - \lambda \tilde{E})\phi\| \, \|\phi\| < \epsilon \end{aligned}$$

for all $n \geqq N$.

Let $E' = I - E$ and let $\bar{M}$ be the factor on $E'H$ obtained by the restriction of $E'ME'$ on $E'H$. We denote by $\bar{A}$ the restriction of $E'AE'$ to $E'H$, and by $\bar{\rho}_n$ and $\bar{\rho}$ the functionals on $\bar{M}$ induced by restricting ρ_n and ρ on $E'ME'$ respectively. Then, since E is the support of ρ, $\bar{\rho} = 0$. Thus we have

$$(1)\quad \begin{aligned} |\rho_n(A)-\rho(A)| &\leqq |\rho_n(EAE)-\rho(EAE)| + |\rho_n(E'AE)| \\ &\quad + |\rho_n(EAE')| + |\rho_n(E'AE')| \\ &\leqq \|\tilde\rho_n-\tilde\rho\|\,\|\tilde A\| + |\rho_n(E'AE)| + |\rho_n(EAE')| \\ &\quad + \|\bar\rho_n\|\,\|\bar A\|. \end{aligned}$$

Here

$$|\rho_n(E'AE)| \leqq \rho_n(E')^{1/2}\rho_n(EA^*AE)^{1/2} \leqq \rho_n(E')^{1/2}\|\tilde\rho_n\|^{1/2}\|(A^*A)^{\sim}\|^{1/2}$$

and

$$|\rho_n(EAE')| \leqq \rho_n(E)^{1/2}\rho_n(E'A^*AE')^{1/2} \leqq \rho_n(E)^{1/2}\|\bar\rho_n\|^{1/2}\|(A^*A)^{-}\|^{1/2}.$$

Since $\rho_n\rightarrow\rho$ weakly, there is a constant K such that $\rho_n(E)^{1/2}\leqq K$ for all n. Thus from (1) we have the following

$$(2)\quad \|\rho_n-\rho\| = \sup_{\|A\|=1} |\rho_n(A)-\rho(A)| \leqq \|\tilde\rho_n-\tilde\rho\| + \rho_n(E')^{1/2}\|\tilde\rho_n\|^{1/2} + K\|\bar\rho_n\|^{1/2} + \|\bar\rho_n\|.$$

By what we have proved, $\|\tilde\rho_n-\tilde\rho\|\rightarrow 0$, and so $\{\|\tilde\rho_n\|\}$ is bounded. Further, $\rho_n(E')^{1/2}\rightarrow\rho(E')^{1/2}=0$ and by Lemma 3, $\|\bar\rho_n\|\rightarrow 0$. Thus it follows from (2) that $\|\rho_n-\rho\|\rightarrow 0$ as $n\rightarrow\infty$.

3. Finally we shall show that a weakly convergent sequence of normal generalized irreducible functionals does not necessarily converge uniformly. An example is obtained by a slight modification of the example given in [1, p. 422]. Let M be a factor of type II_1 and let τ be a (normalized) faithful normal trace of M. Denote by K the completion of the pre-Hilbert space M with inner product $(A, B) =\tau(B^*A)^{1/2}$. Then, by [1, Lemma 5], there is an orthonormal sequence (considered as vectors in K) in M consisting of selfadjoint unitary operators U_n ($n=1, 2, \cdots$) such that $\tau(U_n)=0$. That is, $U_n^2=I$ and $\tau(U_nU_m)=\delta_{nm}$. Define a sequence of projections by $E_n=\frac{1}{2}(I-U_n)$, and put $\rho_n(A)=\tau(E_nA)$ ($A\in M$). Then all ρ_n are generalized irreducible on M by Lemma 2. Since $\{U_n\}$ is orthonormal in K,

$$\sum_n |(A, U_n)|^2 \leqq [[A]] < \infty,$$

where $[[A]]=(A, A)^{1/2}$. Thus $(A, U_n)\rightarrow 0$ as $n\rightarrow\infty$, in other words, $\tau(U_nA)\rightarrow 0$. Namely, $\rho_n(A)=\frac{1}{2}\tau(A-U_nA)\rightarrow\frac{1}{2}\tau(A)$ for each $A\in M$. This means that ρ_n converges weakly to a normal generalized irreducible functional $\frac{1}{2}\tau$. But, having noticed that $\rho_n(U_n)=-\frac{1}{2}$ and $\frac{1}{2}\tau(U_n)=0$,

$$\|\rho_n-\tfrac{1}{2}\tau\| \geqq \tfrac{1}{2} \qquad \text{for all } n.$$

Thus ρ_n does not converge uniformly to $\frac{1}{2}\tau$. Indeed, $\{\rho_n\}$ is not bounded from below by a nonzero normal functional on M. This fact may immediately be verified as follows: $\tau(E_n)=\frac{1}{2}$ for all n and hence

$$\tau(E_1 \cap E_2 \cap \cdots \cap E_n)^2 \leqq (\tfrac{1}{2})^n$$

is proved by induction. Thus there does not exist a nonzero projection F in M such that $F \leqq E_n$ for all n.

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