A NOTE ON THE INDEX OF A G-MANIFOLD[1]

HSU-TUNG KU AND MEI-CHIN KU[2]

1. **Introduction.** By a G-manifold M we mean a compact Lie group G acting differentiably and preserving orientation on an oriented smooth manifold M. The purpose of this paper is to study the index of a $4k$-dimensional G-manifold.

Let M^{4k} be a $4k$-dimensional G-manifold with or without boundary. The cup-product defines a nondegenerate quadratic form f on $H^{2k}(M, \partial M; R)$, where R is the field of real numbers. Let H^+ (resp. H^-) be the maximal subspace on which this form is positive (resp. negative) definite. The subspaces H^+ and H^- are G-modules over R, hence $H^+ - H^- \in RO(G)$. The index of M is defined to be [4]

$$\tau(M) = \dim H^+ - \dim H^-.$$

Now, for any element $g \in G$, the *Atiyah-Singer signature* $\tau(g, M)$ is defined by evaluating the character of $H^+ - H^-$ on g. Hence

$$\tau(g, M) = \text{Trace}\,(g^* \mid H^+) - \text{Trace}\,(g^* \mid H^-),$$

where $g^*: H^{2k}(M, \partial M; R) \to H^{2k}(M, \partial M; R)$.

For any G-module V over R, let

$$V^G = \{v \in V \mid gv = v \quad \text{for all } g \in G\}.$$

We define the G-index $\tau^G(M)$ as follows:

$$\tau^G(M) = \dim(H^+)^G - \dim(H^-)^G.$$

From the definition, we have

PROPOSITION 1.1. *Suppose M is a $4k$-dimensional G-manifold. If G acts trivially on $H^{2k}(M, \partial M; R)$, then $\tau^G(M) = \tau(M)$.*

2. **Main theorems.** The relationship between the index $\tau^G(M)$ and Atiyah-Singer signature $\tau(g, M)$ is the following:

THEOREM 2.1. *Let M be a $4k$-dimensional G-manifold. Then $\tau^G(M) = \int_G \tau(g, M)dg$.*

PROOF. First, we show that

$$\dim(H^\pm)^G = \int_G \text{Trace}\,(g^* \mid H^\pm)dg.$$

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To prove this, let

$$\phi(x) = \int_G g^*(x)dg \quad \text{for } x \in H^{\pm}.$$

Then $\phi \cdot g^* = g^* \cdot \phi = \phi$ for any $g \in G$, and $\phi^2 = \phi$, and so $\operatorname{Im} \phi = (H^{\pm})^G$. Thus $x \in (H^{\pm})^G$ if and only if $\phi(x) = x$. Hence

$$\dim(H^{\pm})^G = \operatorname{Trace}(\phi) = \int_G \operatorname{Trace}(g^* \mid H^{\pm})dg.$$

Combine this result together with the definition of $\tau^G(M)$, we have

$$\tau^G(M) = \int_G \{\operatorname{Trace}(g^* \mid H^+) - \operatorname{Trace}(g^* \mid H^-)\} dg$$
$$= \int_G \tau(g, M)dg.$$

THEOREM 2.2. *Let G be a finite group of order q acting differentiably and preserving orientation on an oriented smooth $4k$-dimensional closed manifold M so that the orbit space M/G has a fundamental class and $\tau(M/G)$ is defined. Then $\tau^G(M) = \tau(M/G)$.*

PROOF. We consider the quadratic form $\bar{f}$ on $H^{2k}(M/G; R)$ defined by cup-product. By [2, p. 38] we have

$$H^*(M/G; R) \overset{\pi^*}{\approx} H^*(M; R)^G,$$

where $\pi: M \to M/G$ is the orbit map. Let f^G be the quadratic form on $H^*(M; R)^G$ defined by cup-product. Then (cf. [3, p. 37])

$$f^G(\pi^*x, \pi^*y) = (\pi^*x \cup \pi^*y)[M] = (x \cup y)\pi_*[M]$$
$$= q(x \cup y)[M/G] = q\bar{f}(x, y),$$

where $[M]$ and $[M/G]$ denote the fundamental classes of M and M/G respectively. Hence the quadratic forms $\bar{f}$ and f^G have the same index, whence $\tau^G(M) = \tau(M/G)$.

THEOREM 2.3. *Let G be a finite group of order q and M be a $4k$-dimensional closed G-manifold. If G acts freely on M, then*

$$\tau(M) = q\tau(M/G).$$

PROOF. We note that the tangent bundle to M is induced by π from the tangent bundle to M/G. Thus π^* maps the Pontrjagin classes $p_i(M/G)$ of M/G onto the Pontrjagin classes $p_i(M)$ of M. Moreover

$\pi_*[M]=q[M/G]$. Hence the Pontrjagin number of M is q times the corresponding Pontrjagin number of M/G. By Hirzebruch index theorem [4], we have $\tau(M)=q\tau(M/G)$. Hence the result follows.

COROLLARY 2.4. *Let G be a finite group of order q acting freely on a $4k$-dimensional closed G-manifold M. Then $\tau(M)=q\tau^G(M)$.*

PROOF. By Theorem 2.2 and Theorem 2.3.

COROLLARY 2.5. *Let G be a nontrivial finite group acting freely on a $4k$-dimensional closed G-manifold M such that G acts trivially on $H^{2k}(M; R)$. Then $\tau(M)=0$. In particular, if G is a compact connected Lie group acting freely on M^{4k}, then $\tau(M)=0$.*

PROOF. By Proposition 1.1 and Corollary 2.4.

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UNIVERSITY OF MASSACHUSSETTS AND
SMITH COLLEGE