

# A NOTE ON THE INDEX OF A $G$ -MANIFOLD<sup>1</sup>

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**1. Introduction.** By a  $G$ -manifold  $M$  we mean a compact Lie group  $G$  acting differentiably and preserving orientation on an oriented smooth manifold  $M$ . The purpose of this paper is to study the index of a  $4k$ -dimensional  $G$ -manifold.

Let  $M^{4k}$  be a  $4k$ -dimensional  $G$ -manifold with or without boundary. The cup-product defines a nondegenerate quadratic form  $f$  on  $H^{2k}(M, \partial M; R)$ , where  $R$  is the field of real numbers. Let  $H^+$  (resp.  $H^-$ ) be the maximal subspace on which this form is positive (resp. negative) definite. The subspaces  $H^+$  and  $H^-$  are  $G$ -modules over  $R$ , hence  $H^+ - H^- \in RO(G)$ . The index of  $M$  is defined to be [4]

$$\tau(M) = \dim H^+ - \dim H^-.$$

Now, for any element  $g \in G$ , the *Atiyah-Singer signature*  $\tau(g, M)$  is defined by evaluating the character of  $H^+ - H^-$  on  $g$ . Hence

$$\tau(g, M) = \text{Trace}(g^* | H^+) - \text{Trace}(g^* | H^-),$$

where  $g^*: H^{2k}(M, \partial M; R) \rightarrow H^{2k}(M, \partial M; R)$ .

For any  $G$ -module  $V$  over  $R$ , let

$$V^g = \{v \in V \mid gv = v \text{ for all } g \in G\}.$$

We define the  $G$ -index  $\tau^g(M)$  as follows:

$$\tau^g(M) = \dim(H^+)^g - \dim(H^-)^g.$$

From the definition, we have

**PROPOSITION 1.1.** *Suppose  $M$  is a  $4k$ -dimensional  $G$ -manifold. If  $G$  acts trivially on  $H^{2k}(M, \partial M; R)$ , then  $\tau^g(M) = \tau(M)$ .*

**2. Main theorems.** The relationship between the index  $\tau^g(M)$  and Atiyah-Singer signature  $\tau(g, M)$  is the following:

**THEOREM 2.1.** *Let  $M$  be a  $4k$ -dimensional  $G$ -manifold. Then  $\tau^g(M) = \int_G \tau(g, M) dg$ .*

**PROOF.** First, we show that

$$\dim(H^\pm)^g = \int_G \text{Trace}(g^* | H^\pm) dg.$$

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To prove this, let

$$\phi(x) = \int_G g^*(x)dg \quad \text{for } x \in H^\pm.$$

Then  $\phi \cdot g^* = g^* \cdot \phi = \phi$  for any  $g \in G$ , and  $\phi^2 = \phi$ , and so  $\text{Im } \phi = (H^\pm)^G$ . Thus  $x \in (H^\pm)^G$  if and only if  $\phi(x) = x$ . Hence

$$\dim(H^\pm)^G = \text{Trace}(\phi) = \int_G \text{Trace}(g^* | H^\pm)dg.$$

Combine this result together with the definition of  $\tau^G(M)$ , we have

$$\begin{aligned} \tau^G(M) &= \int_G \{ \text{Trace}(g^* | H^+) - \text{Trace}(g^* | H^-) \} dg \\ &= \int_G \tau(g, M) dg. \end{aligned}$$

**THEOREM 2.2.** *Let  $G$  be a finite group of order  $q$  acting differentiably and preserving orientation on an oriented smooth  $4k$ -dimensional closed manifold  $M$  so that the orbit space  $M/G$  has a fundamental class and  $\tau(M/G)$  is defined. Then  $\tau^G(M) = \tau(M/G)$ .*

**PROOF.** We consider the quadratic form  $\bar{f}$  on  $H^{2k}(M/G; R)$  defined by cup-product. By [2, p. 38] we have

$$H^*(M/G; R) \overset{\pi^*}{\approx} H^*(M; R)^G,$$

where  $\pi: M \rightarrow M/G$  is the orbit map. Let  $f^G$  be the quadratic form on  $H^*(M; R)^G$  defined by cup-product. Then (cf. [3, p. 37])

$$\begin{aligned} f^G(\pi^*x, \pi^*y) &= (\pi^*x \cup \pi^*y)[M] = (x \cup y)\pi_*[M] \\ &= q(x \cup y)[M/G] = q\bar{f}(x, y), \end{aligned}$$

where  $[M]$  and  $[M/G]$  denote the fundamental classes of  $M$  and  $M/G$  respectively. Hence the quadratic forms  $\bar{f}$  and  $f^G$  have the same index, whence  $\tau^G(M) = \tau(M/G)$ .

**THEOREM 2.3.** *Let  $G$  be a finite group of order  $q$  and  $M$  be a  $4k$ -dimensional closed  $G$ -manifold. If  $G$  acts freely on  $M$ , then*

$$\tau(M) = q\tau(M/G).$$

**PROOF.** We note that the tangent bundle to  $M$  is induced by  $\pi$  from the tangent bundle to  $M/G$ . Thus  $\pi^*$  maps the Pontrjagin classes  $p_i(M/G)$  of  $M/G$  onto the Pontrjagin classes  $p_i(M)$  of  $M$ . Moreover

$\pi_*[M] = q[M/G]$ . Hence the Pontrjagin number of  $M$  is  $q$  times the corresponding Pontrjagin number of  $M/G$ . By Hirzebruch index theorem [4], we have  $\tau(M) = q\tau(M/G)$ . Hence the result follows.

**COROLLARY 2.4.** *Let  $G$  be a finite group of order  $q$  acting freely on a  $4k$ -dimensional closed  $G$ -manifold  $M$ . Then  $\tau(M) = q\tau^G(M)$ .*

**PROOF.** By Theorem 2.2 and Theorem 2.3.

**COROLLARY 2.5.** *Let  $G$  be a nontrivial finite group acting freely on a  $4k$ -dimensional closed  $G$ -manifold  $M$  such that  $G$  acts trivially on  $H^{2k}(M; R)$ . Then  $\tau(M) = 0$ . In particular, if  $G$  is a compact connected Lie group acting freely on  $M^{4k}$ , then  $\tau(M) = 0$ .*

**PROOF.** By Proposition 1.1 and Corollary 2.4.

#### REFERENCES

1. M. F. Atiyah and I. M. Singer, *The index of elliptic operators*. III, Ann. of Math. **87** (1968), 546–604.
2. A. Borel et al., *Seminar on transformation groups*, Ann. of Math. Studies, No. 46, Princeton Univ. Press, Princeton, 1960.
3. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Springer-Verlag, New York, 1964.
4. F. Hirzebruch, *Topological methods in algebraic geometry*, Springer-Verlag, New York, 1966.
5. Hsu-Tung Ku and Mei-Chin Ku, *On the  $G$ -index of a  $G$ -manifold*, Mimeographed, Institute for Advanced Study, Princeton, N. J., 1967.

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