ANALYTIC CONTINUATION OF HOLOMORPHIC FUNCTIONS WITH VALUES IN A LOCALLY CONVEX SPACE

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Horváth [3] has announced a result generalizing the result of Gelfand and Shilov [2] on analytic continuation of holomorphic functions with values in a locally convex space. In this paper we shall present a generalization of these results which permits one to prove the existence of strong holomorphic extensions from the existence of weak or weak* holomorphic extensions. For more general results see [5].

1. **Definition of a norming triple.** Let Y be a complex linear space and p a seminorm on it. Assume that there is given a complete seminormed space $(Y_p, \|\ \|_p)$ and a bilinear functional $(\ ,\ )_p$ from the space $Y\times Y_p$ into the space C of complex numbers. We shall say that the triple $(Y_p, \|\ \|_p, (\ ,\ )_p)$ is a norming triple for the seminorm p if

$$p(y) = \sup\{ |(y, z)_p| : z \in Y_p, \|z\|_p \leqq 1\} \qquad \text{for all } y \in Y.$$

EXAMPLE 1. Let p be a seminorm on the space Y. Denote by Y_p the family of all linear functionals z on the space Y such that $|z(y)| \leqq cp(y)$ for all $y\in Y$ and some $c>0$. Define the norm of the functional by $\|z\|_p=\inf c$, where the infimum is taken over all constants satisfying the previous condition. Define the bilinear functional by means of the formula $(y, z)_p=z(y)$ for all $y\in Y$, $z\in Y_p$.

Using the generalization of the Hahn-Banach Theorem to the case of complex linear spaces one can easily prove that the triple $(Y_p, \|\ \|_p, (\ ,\ )_p)$ is norming for the seminorm p.

EXAMPLE 2. Let E be a sequentially complete, complex, locally convex space. Assume that the topology on it is generated by the family $Q=\{q\}$ of seminorms. Consider the strong dual $Y=E'$. One may assume that the topology on the space is generated by the family of seminorms p given by means of the formula:

$$p(y) = \sup\{ |y(z)| : z \in B\}$$

for all $y\in Y$, where B runs through all bounded sets B of the space E, which can be represented in the form $B=\{z\in E: q(z)\leqq c_q$ for all $q\in Q\}$, c_q being a family of positive constants.

Define an extended seminorm by means of the formula

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$\|z\|_p = \sup\{c_q^{-1}q(z) : q \in Q\}$ for $z \in E$. Let $Y_p = \{z \in E : \|z\|_p < \infty\}$. Notice that the space $(Y_p, \| \|_p)$ is a complete seminormed space and $B = \{z \in Y_p : \|z\|_p \leqq 1\}$. Define the bilinear functional by means of the formula $(y, z)_p = y(z)$ for all $y \in Y$, $z \in Y_p$. It is easy to see that the triple $(Y_p, \| \|_p, (,)_p)$ is a norming triple for the seminorm p.

2. **Definition of a holomorphic function.** Let D be a domain in the complex plane C. Let Y be a sequentially complete, complex, locally convex Hausdorff space. Assume that f is a function from the domain D into the space Y. We shall say that the function f is holomorphic if for every point $x_0 \in D$ there exists a positive radius r such that $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ if $|x - x_0| < r$, where $a_n \in Y$ and

$$\{x \in C : |x - x_0| < r\} \subset D.$$

Assume that p is a seminorm on the linear space Y and that $(Y_p, \| \|_p, (,)_p)$ is a norming triple for the seminorm. It is easy to prove from the definitions of a norming triple the inequality $|(y, z)_p| \leqq p(y)\|z\|_p$ for all $y \in Y$, $z \in Y_p$. This implies that the functional $(\cdot, z)_p$ is continuous and therefore the scalar function h defined by $h(x) = (f(x), z)_p$ for $x \in D$ is holomorphic in the classical sense, if the function f is holomorphic from the set D into the locally convex space Y.

One can prove the usual formula for the coefficients in the Taylor expansion of the holomorphic function:

$$a_n = f^{(n)}(x_0)/n! \qquad \text{for } n = 0, 1, 2, \cdots.$$

Assume that D is an open domain and D_1 is another open domain such that $D \subset D_1$.

THEOREM. *Let Y be a complex sequentially complete locally convex Hausdorff space with topology generated by a family $P = \{p\}$ of seminorms. To every seminorm p let correspond a norming triple $(Y_p, \| \|_p, (,)_p)$. Let f be a holomorphic function from the domain D into the space Y and assume that for every seminorm p and every $z \in Y_p$ the holomorphic function $x \mapsto (f(x), z)_p$ has an extension to a holomorphic function on the domain D_1. Then there exists a holomorphic function f_1 from the domain D_1 into Y such that $f_1(x) = f(x)$ for all $x \in D$.*

PROOF. The proof of the theorem is based on the folllowing lemmas.

LEMMA 1. *Let p be a fixed seminorm from the the family P. Let $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ if $|x - x_0| < r_0$. Assume that the function*

$x \mapsto (f(x), z)_p$ has a holomorphic extension onto a disc $|x-x_0|<r$, where $r_0<r$. Then $\lim\sup(p(a_n))^{1/n}r \leqq 1$.

PROOF OF LEMMA 1. Since $(\cdot, z)_p$ is a linear continuous functional for every z in Y_p and

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \qquad \text{if } |x - x_0| < r_0,$$

we have

$$(f(x), z)_p = \sum_{n=0}^{\infty} (a_n, z)_p(x - x_0)^n \qquad \text{if } |x - x_0| < r_0.$$

Since the function on the left side of the last equality has a holomorphic extension onto the circle of radius r, we get that the series $\sum_{n=0}^{\infty} (a_n, z)_p (x-x_0)^n$ is convergent at every point of the disc $|x-x_0|<r$.

Take any number $0<a<r$ and consider the sequence of sets

$$F_m = \{z \in Y_p: |(a_n, z)_p| a^n \leqq m \text{ for all } n\}.$$

It is easy to prove that the sets F_m are closed, circled, and contain the element zero. Moreover, we have the equality $Y_p = \bigcup_{m=1}^{\infty} F_m$. The space $(Y_p, \| \|_p)$ being a complete seminormed space implies that there exists a positive radius s and a positive integer m such that $z \in F_m$ if $\|z\|_p \leqq s$. This implies that there exists a positive constant $s(a)$ such that $|(a_n, z)_p| a^n \leqq s(a)\|z\|_p$ for $z \in Y_p$, $n=0, 1, \cdots$. Taking the supremum over all elements z such that $\|z\|_p \leqq 1$ we get $p(a_n)\, a^n \leqq s(a)$ for all n. Thus

$$\lim\sup(p(a_n))^{1/n}a \leqq 1$$

for all $a<r$. Passing to the limit in the last inequality when $a \to r$ we obtain the conclusion of Lemma 1.

LEMMA 2. *Let $a_n \in Y$ and assume the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges if $|x-x_0|<r_0$. If for every seminorm $p \in P$ and every $z \in Y_p$ the function $x \mapsto (f(x), z)_p$ has a holomorphic extension onto the disc $|x-x_0| <r$, where $r_0<r$, then the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges almost uniformly in the disc $|x-x_0|<r$, in the topology of the space Y to a continuous function f_1.*

PROOF OF LEMMA 2. Take any point x from the disc $|x-x_0|<r$. It follows from Lemma 1 that the series $\sum p(a_n)|x-x_0|^n$ is convergent for every seminorm $p \in P$. This implies that the sequence $s_n(x) = \sum_{j=0}^{n} a_j(x-x_0)^j$ is Cauchy in the space Y. The space being

sequentially complete, there exists an element $f_1(x) \in Y$ such that

$$f_1(x) = \lim s_n(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

Notice that in the disc $|x-x_0| \leqq a<r$ we have the uniform estimate

$$p(f_1(x) - s_n(x)) \leqq \sum_{j>n} p(a_j)a^j.$$

This implies that the function s_n converges almost uniformly to the function f_1. Since the functions s_n are continuous we get the continuity of the function f_1.

LEMMA 3. *Let f be a continuous function from a domain D into the space Y. If for every regular closed curve C contained with its interior in D the line integral $\int_C f(x)\,dx$ is zero, then the function f is holomorphic in D.*

The proof of Lemma 3 is obvious.

LEMMA 4. *Let f be a holomorphic function from a disc $|x-x_0| <r_0$, into the space Y. If for every seminorm $p\in P$ and every $z\in Y_p$ the function $x\mapsto(f(x), z)_p$ has a holomorphic extension onto the disc $|x-x_0| <r$, where $r_0<r$, then there exists a holomorphic function f_1 from the disc $|x-x_0| <r$ into the space Y such that $f_1(x) =f(x)$ if $|x-x_0| <r_0$.*

The proof of the lemma follows from Lemmas 1, 2, and 3.

Take a point $x_0\in D$ and a point $x\in D_1$. By a chain joining the points x_0 and x we shall understand a collection of open discs $S_0, \cdots, S_n$ contained in the domain D_1 and such that the center of the disc S_0 is the point x_0 and $S_0\subset D$, the center of the disc S_j is in the disc S_{j-1} for all j, and $x\in S_n$.

LEMMA 5. *Let f satisfy the assumptions of the theorem. Let $S_0, \cdots, S_n$ be a chain joining the point $x_0\in D$ with the point $x\in D_1$. Then f has a holomorphic continuation along the chain. If $S_0^1, \cdots, S_m^1$ is another chain joining the points x_0 and x then the extension of the function f along the first chain yields the same function on $S_n\cap S_m^1$ as the extension along the second chain.*

PROOF OF LEMMA 5. Denote by g_1 the holomorphic function on S_n being the holomorphic extension of the function f along the first chain. Let g_2 be the holomorphic function on the disc S_m^1 being the holomorphic extension of the function along the second chain. Since the scalar function $x\mapsto(f(x), z)_p$ has a unique extension from the domain D onto the domain D_1, we get $(g_1(x), z)_p = (g_2(x), z)_p$ if $x \in S_n \cap S_m^1$. The element $z \in Y_p$ being arbitrary, we get $p(g_1(x) - g_2(x)) = 0$ for all $p \in P$. Since the space Y is Hausdorff, we conclude $g_1(x) = g_2(x)$ on $S_n \cap S_m^1$.

PROOF OF THE THEOREM. Take any point $x \in D_1$ and consider a fixed point $x_0 \in D$. Define a function f_1 by means of the formula $f_1(t) = g(t)$ for all $t \in S$, where g represents a holomorphic function on the disc, $S_n = S$ being a holomorphic extension along the chain $S_0, \cdots, S_n$ joining the points x_0, x. It follows from Lemma 5 that the function f_1 is well defined. Moreover the function represents a holomorphic extension of the function f from the domain D onto the domain D_1.

The following corollary represents a generalization of the result due to Horváth [3]. We have removed the assumption that the space Z is Hausdorff.

COROLLARY 1. *Let Z be a sequentially complete, barreled, complex, locally convex space and let $Y = Z'$ be the strong dual, i.e. with topology of uniform convergence on all bounded sets of Z. Let $f: D \to Y$ be a holomorphic function such that each of the scalar functions $x \mapsto f(x)(z)$, $z \in Z$, has a holomorphic extension onto the set D_1. Then there exists a holomorphic function $f_1: D_1 \to Y$ extending the function f.*

The proof of the corollary follows immediately from Example 2. Indeed notice that the topology on the space Y can be introduced by means of the seminorms

$$p_c(y) = \sup\{ | y(z) | : \|z\|_c \leq 1\},$$

where

$$\|z\|_c = \sup\{ c_q^{-1} q(z) : q \in Q\} \qquad \text{for } z \in Z,$$

and c denotes any function $q \mapsto c_q > 0$ for all $q \in Q$. The space Y endowed with the family of seminorms $\{p_c\}$ is a complete, Hausdorff, complex, locally convex space [4]. Notice that the triple $(Y_c, \| \|_c, (,)_c)$ is norming for the functional p_c where $Y_c = \{z \in Z : \|z\|_c < \infty\}$, and $(y, z)_c = y(z)$ for all $y \in Y$, $z \in Y_c$ according to Example 2. Thus all the assumptions of the theorem are satisfied.

COROLLARY 2. *Let Z be a complete seminormed space and let $Y = Z'$ be its strong dual. Let f be a holomorphic function from D into Y such that for every $z \in Z$ the scalar function $x \mapsto f(x)(z)$ has a holomorphic extension onto the set D_1. Then there exists a holomorphic function f_1 from D_1 into Y extending the function f.*

Notice that this corollary is a particular case of Corollary 1.

COROLLARY 3. *Let Y be a sequentially complete, Hausdorff, complex, locally convex space. Let $f: D \mapsto Y$ be a holomorphic function such that for every linear continuous functional $y' \in Y'$ the scalar function $x \mapsto y'f(x)$ has a holomorphic extension onto the set D_1. Then there exists a holomorphic function $f_1: D_1 \mapsto Y$ extending the function f.*

The proof of the corollary follows from Example 1.

COROLLARY 4. *Let Y be a complex Banach space. Let f be a holomorphic function from the domain D into the space Y such that for every linear continuous functional $y' \in Y'$ the scalar function $x \mapsto y'f(x)$ has a holomorphic extension onto the domain D_1. Then there exists a holomorphic function $f_1: D_1 \mapsto Y$ extending the function f.*

This corollary is a particular case of the preceding one.

Corollary 4 represents a generalization of the result due to Horváth [**3**, Theorem 2], proven for the case of the space $Y = c_0$ of sequences convergent to zero.

Compare the following with [**6**], [**7**].

Let Q be a compact set in the space R^n. Assume that the interior of the set Q is dense in Q. Let $C(Q)$, $L(Q)$, $H(Q, D)$ be, respectively, the space of complex continuous functions on Q, the space of Lebesgue summable functions on Q, the space of all continuous functions f from $Q \times D$ into the space C of complex numbers such that for every $q \in Q$ the function $f(q, \cdot)$ is holomorphic.

COROLLARY 5. *Let $f \in H(Q, D)$ and assume that for every function $g \in L(Q)$ the function $\int g(q) f(q, \cdot) dq$ has a holomorphic extension onto the set D_1. Then there exists a function $f_1 \in H(Q, D_1)$ extending the function f.*

Notice that the space $H(Q, D)$ can be considered as the space of holomorphic functions from D into $C(Q)$ and that the triple $(L(Q), \| \ \|_L, (\ , \ )_L)$, where $\|f\|_L = \int |f(q)| dq$ for all $f \in L(Q)$, and $(f, g)_L = \int f(q) g(q) dq$ for all $g \in L(Q)$ and $f \in C(Q)$, is norming for the norm in the space $C(Q)$. This follows from [**1**, Theorem 5, p. 289].

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