SOME STEADY STATE PROPERTIES OF $(\int_0^x f(t)dt)/f(x)$

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1. **Introduction.** In many physical situations described by a differential equation of the form

$$(1) \qquad y' + a(x)y = b(x), \qquad 0 \leqq x < \infty,$$

it is desirable that a solutlon y be found with the property that, for large values of x, y behaves in some sense like a periodic function [**1**]. We observe that the solution of (1) can easily be placed in the form $y = b(x)(\int_0^x f(t)dt)/f(x)$ whenever $b(x) \neq 0$ for any x and $f(x) = b(x)\exp[\int a(x)]$. The solution y depends strongly upon the properties of $(\int_0^x f(t)dt)/f(x)$. We define τf by the correspondence

$$\tau f(x) = (\int_0^x f(t)dt)/f(x)$$

where $f \in \mathfrak{H} = \{f : f$ is a strictly positive continuous function defined on the half ray $[0, \infty\rangle\}$. Under what conditions will τf behave like a periodic function for large values of x? In this paper we supply a partial answer to this question via some necessary (Theorem 2) and sufficient (Theorems 1, 3, 4) conditions. In Theorem 5, we establish that under suitable hypotheses the derivative of the steady state of τf is equal to the steady state of the derivative of τf.

2. **Preliminary notions.** We shall need the following definitions amd properties of steady state functions. Further, we fix some notation.

If F is a function on $[0, \infty\rangle$ and G is a function whose domain contains $[0, \infty\rangle$, then G is defined to be a steady state of F if for every number $\epsilon > 0$, there exists a number $T > 0$ such that for $t > T$, $|F(t) - G(t)| < \epsilon$. Lopez [**3**] proved that if G_1 and G_2 are continuous and periodic functions which admit the same period, where both G_1 and G_2 are steady states of F, then $G_1 = G_2$. This theorem is also true when G_1 and G_2 are constant functions. Further, it can be shown that whenever F is a continuous function on $[0, \infty\rangle$, and if G and K are periodic steady states of F on $[0, \infty\rangle$, then $G = K$. Thus a periodic steady state is unique.

The following lemmas can be derived as immediate consequences of the definition of steady state.

Received by the editors July 9, 1968.

LEMMA 1. *If G is a continuous real-valued function which admits period h, and G is a steady state of F on $[0, \infty\rangle$, then the sequence $\{F(t+nh)\}_{n=0}^{\infty}$ approaches $G(t)$ uniformly on $[0, \infty\rangle$.*

LEMMA 2. *If F is a continuous function on $[0, \infty\rangle$, and if there exists a positive number h such that the sequence $\{F(x+nh)\}_{n=0}^{\infty}$ converges uniformly on $[0, h]$, then a steady state G of F exists which admits period h and for every $x\in[0, \infty\rangle$, $G(x)=\lim_{n\to\infty} F(x+nh)$.*

THEOREM 1. *If f and $g\in\mathfrak{H}$, and $f(x)=g(x)$ for $x\geqq T$, $\lim f(x)=\infty$, then F is a steady state of τf if and only if F is a steady state of τg.*

PROOF.

$$\tau g(x) = \tau f(x) + \frac{\int_0^T [g(t) - f(t)]dt}{f(x)}, \qquad x \geqq T.$$

Hence, $\lim_{x\to\infty} [\tau g(x) - \tau f(x)] = 0$.

LEMMA 3. *A necessary condition for $\tau f(x) \geqq s$ for $x \geqq T$, s is a constant, is that $|f(x)| \leqq c_1 \exp(c_2 x)$, where c_1 and c_2 will be defined below.*

PROOF. Consider the equation $[\tau f(x)]f(x) = \int_0^x f(t)dt$. Then $([\tau f(x)]f(x))' = f(x)$ or $[(\tau f)f]'/(\tau f)f = 1/\tau f$. Thus

$$f(x) = \frac{c}{\tau f(x)} \exp\left(\int_t^x \frac{dt}{\tau f(t)}\right)$$

or

$$|f(x)| \leqq \frac{c}{s} \exp\frac{(x - T)}{s} \qquad \text{for } x \geqq T.$$

Thus, $c_1 = (c/s)\exp(-Ts^{-1})$ and $c_2 = s^{-1}$. Observe that if $|f(x)| \leqq c_1 \cdot \exp(c_2 x)$ is not satisfied for $x \geqq T$, τf may have a periodic steady state which takes on zero values as evidenced by $f(x) = e^x e^{e^x}$.

LEMMA 4. *A necessary condition that τf itself be its own periodic steady state of period h for sufficiently large x is that $f(x) = cf(x+h)$ for some constant c, $x \geqq T$.*

PROOF. For $x \geqq T$, we wish

$$\frac{\int_0^{x+h} f(t)dt}{f(x+h)} - \frac{\int_0^x f(t)dt}{f(x)} = 0.$$

Let $G(x)=\int_0^x f(t)dt$. Then we must have $G'(x)G(x+h)-G'(x+h)G(x)=0$, or $(d/dx)[G(x)/G(x+h)]=0$ or $G(x)=cG(x+h)$ for some constant c. Taking derivatives, we get $f(x)=cf(x+h)$ for $x\geqq T$.

This condition is not sufficient, as is shown by $f(x)=e^x$. An example of a function $f(x)$ such that τf is its own periodic steady state is

$$\begin{aligned} f(x) &= e^x, & 0 \leqq x \leqq \log 2, \\ &= \tfrac{1}{2}e^{2x}, & \log 2 < x. \end{aligned}$$

If $p(x)$ is a nonnegative, continuous, periodic function, then there is a function f for which the steady state of τf is $p(x)$. Define f by the correspondence

$$f(x) = \frac{\exp\left(\int_0^x \frac{dt}{p(t) + 1/(t+1)}\right)}{p(x) + 1/(x+1)} .$$

Throughout this paper, σF shall denote the unique periodic steady state of F; in particular, $\sigma\tau f$ is the unique steady state of τf. Further,
$\mathfrak{F} = \{f\in\mathfrak{H}: \sigma\tau f \text{ exists}\}$,
$\mathfrak{F}h = \{f\in\mathfrak{F}: \sigma\tau f \text{ admits period } h\}$.

Main theorems.

THEOREM 2. *If $f\in\mathfrak{F}$, then*
(i) *τf is bounded on* $[0, \infty\rangle$;
(ii) *there exists $m>0$ such that* $m=\inf\{f(x): 0\leqq x<\infty\}$;
(iii) *f is unbounded above*;
(iv) $f(x)\geqq mx^k/k!M^k$, $k=0, 1, 2, \cdots, x\in[0, \infty\rangle$;
(v) $f(x)\geqq((r-1)/r)m[\exp(x/rM)]$, $r>1$.
In (iv) *and* (v) *above as well as in the proof which follows, m and M denote positive lower and upper bounds of f and τf, respectively.*

PROOF. Trivially, (i) follows from the definition of $\mathfrak{F}$. The proof of (ii) will be given below in some detail. By a straightforward argument using the definition of τf, (i) and (ii), we derive $mx\leqq\int_0^x f(t)dt\leqq Mf(x)$, which implies (iii). By induction on k, (iv) is established. To derive (v), we observe from (iv) that $f(x)/r^k\geqq m[x/(rM)]^k/k!$. Therefore,

$$\sum_{k=0}^{\infty}(1/r)^k f(x) \geqq \sum_{k=0}^{\infty}\frac{m[x/(rM)]^k}{k!} = m\cdot\exp[x/(rM)].$$

PROOF OF (ii). Assume the contrary, then sequences $\{x_n\}$ and

$\{f(x_n)\}$ exist for which $x_n < x_{n+1}$, $x_0 = 0$ and $f(x_{n+1}) < f(x_n)$, so that

$$\lim_{n\to\infty} f(x_n) = 0. \tag{2}$$

Let $g(x_n) = \int_0^{x_n} f(t)dt$. Then $\{g(x_n)\}$ is a strictly increasing sequence. From (i), there exists $M > 0$ so that $\tau f(x_n) = g(x_n)/f(x_n) < M$ for each n. Further, $f \in \mathfrak{H}$ and the sequence $\{f(x_n)\}$ is strictly decreasing implies $g(x_n) < Mf(x_n) < Mf(x_0) = Mf(0)$.

Thus, the sequences $\{\tau f(x_n)\}$ and $\{g(x_n)\}$ are increasing and bounded above by M and $Mf(0)$, respectively. Therefore they converge to positive real numbers. However, $1/f(x_n) = \tau f(x_n)/g(x_n)$, which implies that $\lim_{n\to\infty} (1/f(x_n))$ is finite, thereby contradicting (2).

THEOREM 3. *If $f \in \mathfrak{F}h$, $|g(x)| \leqq Nx^k$ on $[0, \infty\rangle$ for some nonnegative integer k and some positive real number N, $f+g \in \mathfrak{H}$, then $f+g \in \mathfrak{F}h$ and $\sigma\tau(f+g) = \sigma\tau f$.*

PROOF. By definition,

$$\tau(f + g)(x + nh) = \frac{\tau f(x + nh)}{1 + g(x + nh)/f(x + nh)} + \frac{\int_0^{x+nh} g(t)dt}{f(x + nh) + g(x + nh)}.$$

The sequence $\{g(x+nh)/f(x+nh)\}$ converges uniformly to zero on $[0, h]$. This can be reasoned by applying Theorem 2(iv) and the hypothesis $|g(x)| \leqq Nx^k$. From Lemma 1, $\tau f(x+nh)$ converges to $\sigma\tau f(x)$ uniformly on $[0, \infty\rangle$.

If $x \in [0, h]$ and $n+1 > h^{-1}(NM^{k+2}(k+2)!/m)^{1/2}$, where M, m and k are defined as in Theorem 2(iv), then

$$c = 1 - \frac{NM^{k+2}(k + 2)!}{m(n + 1)^2h^2} > 0.$$

Further,

$$\left|\frac{\int_0^{x+nh} g(t)dt}{f(x + nh) + g(x + nh)}\right| \leqq \frac{NM^{k+2}(k + 2)!}{cmh(k + 1)(n + 1)}.$$

Thus $\tau(f+g)(x+nh)$ is the sum of two uniformly converging sequences on $[0, h]$. The theorem follows from Lemma 2.

The following theorem is an immediate consequence of Theorems 1 and 3.

THEOREM 4. *If $g \in \mathfrak{H}$, $f \in \mathfrak{F}h$ and there exist nonnegative real numbers m and T, and a nonnegative integer k such that for every $x \geqq T$, $|g(x) - f(x)| \leqq mx^k$, then $g \in \mathfrak{F}h$ and $\sigma\tau g = \sigma\tau f$.*

THEOREM 5. *If $f \in \mathfrak{F}h$, f is a differentiable function on $[0, \infty\rangle$, f' is continuous on $[0, \infty\rangle$ and there exists a nonnegative continuous periodic real-valued function, p, with period h such that $\sigma(f'/f) = p$, then $\sigma\tau f$ is a differentiable function and $(\sigma\tau f)' = \sigma[(\tau f)']$.*

PROOF. Clearly, τf is a differentiable function whose derivative is given by

$$(\tau f)' = 1 - f'(\tau f)/f.$$

Thus

$$\sigma[(\tau f)'] = 1 - p(\sigma\tau f) \tag{3}$$

follows immediately from the hypotheses and definition of steady state.

If n is a positive integer, $x \in [0, \infty\rangle$ and k is an increment,

$$\begin{aligned} \tau f(x + nh + k) - \tau f(x + nh) &= k - \int_0^k p(v + x)\sigma\tau f(v + x)dv \\ &+ \int_0^k \left[p(v + x)\sigma\tau f(v + x) - \frac{f'(v + x + nh)}{f(v + x + nh)} \tau f(v + x + nh) \right] dv. \end{aligned} \tag{4}$$

Since $\sigma\tau f$ and f'/f are bounded on $[0, \infty\rangle$, and from Lemma 1, $f'(x+nh)/f(x+nh)$ and $\tau f(x+nh)$ converge uniformly to $p(x)$ and $\sigma\tau f(x)$, respectively, on $[0, \infty\rangle$, then

$$\int_0^k \left[p(v + x)\sigma\tau f(v + x) - \frac{f'(v + x + nh)}{f(v + x + nh)} \tau f(v + x + nh) \right] dv$$

converges uniformly to zero on $[0, \infty\rangle$, as $n \to \infty$. On applying Lemma 1 to (4), we get

$$\sigma\tau f(x + k) - \sigma\tau f(x) = k - \int_0^k p(v + x)\sigma\tau f(v + x)dv.$$

If we let $k \to 0$, we derive

$$(\sigma\tau f)'(x) = 1 - p(x)\sigma\tau f(x). \tag{5}$$

The theorem follows from (3) and (5).

The author acknowledges with appreciation the numerous suggestions of Professor Don E. Edmondson, The University of Texas. Also, the author is indebted to Professor H. I. Freedman, The University of Alberta, Edmonton, Canada, for Lemmas 3 and 4. Preparation of this paper was supported in part by the National Science Foundation (NSF-Gy 3901) while the author was at the University of Oklahoma during the summer of 1967.

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