

ON THE IRREDUCIBILITY OF WEIGHTED SHIFTS

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Let $\{\phi_n\}$ ($n=0, 1, \dots$) be an orthonormal basis of a Hilbert space H and let $\{\alpha_n\}$ be a bounded sequence of complex scalars. Then a unilateral weighted shift A with weights α_n is the operator on H defined by

$$A\phi_n = \alpha_n\phi_{n+1} \quad (n = 0, 1, \dots),$$

and its adjoint is given by

$$A^*\phi_0 = 0 \quad \text{and} \quad A^*\phi_n = \bar{\alpha}_{n-1}\phi_{n-1} \quad (n = 1, 2, \dots).$$

It is known that a unilateral shift, more generally, a unilateral weighted shift with nonzero weights is irreducible, i.e., it has no non-trivial reducing subspace. In the present paper we shall show the *strong* irreducibility of a unilateral weighted shift with nonzero weights in the following sense:

THEOREM 1. *Every operator on a Hilbert space which is similar to a unilateral weighted shift with nonzero weights is irreducible.*

As a striking application of this result, we shall give a satisfactory answer to the question: which unilateral weighted shifts with nonzero weights are spectral in the sense of Dunford? Indeed, it will be shown that such a weighted shift can be spectral only if it is quasi-nilpotent.

THEOREM 2. *A unilateral weighted shift with nonzero weights is a spectral operator if and only if it is quasi-nilpotent.*

If A is a unilateral weighted shift with weights α_n , then a straightforward computation shows

$$\|A^k\| = \sup_n \left| \prod_{i=0}^{k-1} \alpha_{n+i} \right| \quad (k = 1, 2, \dots),$$

and hence the spectral radius $\gamma(A)$ is equal to $\lim_k \sup_n \left| \prod_{i=0}^{k-1} \alpha_{n+i} \right|^{1/k}$ (cf. [2]). It follows that A is quasi-nilpotent if and only if

$$\sup_n \left| \prod_{i=0}^{k-1} \alpha_{n+i} \right|^{1/k} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

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Now let us consider a nonzero monotone shift A with weights α_n (i.e., $|\alpha_n| \leq |\alpha_{n+1}|$ for each n). Stampfli [3] has shown that such an A is not spectral. Obviously this fact may be derived from the above theorem. In fact, let N denote the smallest index such that $\alpha_N \neq 0$. Then, having noticed that the restriction of A to the subspace generated by $\phi_0, \phi_1, \dots, \phi_{N-1}$ is zero, the subspace K generated by $\phi_N, \phi_{N+1}, \dots$ is a reducing subspace of A and the restriction $A|_K$ of A to K is a unilateral weighted shift on K with nonzero weights. Thus, by what we have observed above, $A|_K$ is not quasi-nilpotent, that is, it is not spectral by Theorem 2. It follows that A itself is not spectral. More generally, a similar argument shows the following

COROLLARY. *Let A be a unilateral weighted shift with weights α_n . If $\alpha_0 = \alpha_1 = \dots = \alpha_N = 0$ and there is a positive constant c such that $|\alpha_n| \geq c$ for all $n > N$, then A is not a spectral operator.*

1. In what follows, A always means a unilateral weighted shift with nonzero weights α_n associated with an orthonormal basis $\{\phi_n\}$ ($n = 0, 1, \dots$) of a Hilbert space H . The proof of Theorem 1 is based on a specific property of the system of invariant subspaces of A^* .

LEMMA. *The Hilbert space H can not be expressed as an algebraic direct sum of two nontrivial invariant subspaces of A^* .*

PROOF. Suppose that H is decomposed in the algebraic direct sum of nontrivial invariant subspaces \mathfrak{M} and \mathfrak{N} of A^* . Since $\mathfrak{M} \cap \mathfrak{N} = \{0\}$, we may assume that at least one of the subspaces \mathfrak{M} and \mathfrak{N} does not contain the vector ϕ_0 . Let $\phi_0 \notin \mathfrak{M}$. Then $\phi_0 = \psi_1 + \psi_2$, where $\psi_1 \in \mathfrak{M}$ and $\psi_2 \in \mathfrak{N}$. Consider the Fourier expansions of ψ_1 and ψ_2 :

$$\psi_1 = \sum_n \lambda_n \phi_n,$$

where $\lambda_n \neq 0$ for some $n \geq 1$ (recall that $\phi_0 \notin \mathfrak{M}$);

$$\psi_2 = \sum_n \mu_n \phi_n.$$

Then $\phi_0 = \sum_n (\lambda_n + \mu_n) \phi_n$ implies that $\mu_0 = 1 - \lambda_0$ and $\mu_n = -\lambda_n$ ($n \geq 1$). Thus $-\psi_2 = (\lambda_0 - 1)\phi_0 + \sum_{n \geq 1} \lambda_n \phi_n$. Since \mathfrak{N} is invariant under A^* , we have

$$A^*(-\psi_2) = \sum_{n \geq 1} \lambda_n \bar{\alpha}_{n-1} \phi_{n-1} \in \mathfrak{N}.$$

On the other hand, we have

$$A^*\psi_1 = \sum_{n \geq 1} \lambda_n \bar{\alpha}_{n-1} \phi_{n-1} \in \mathfrak{M}.$$

Consequently, $\psi \equiv \sum_{n \geq 1} \lambda_n \bar{\alpha}_{n-1} \phi_{n-1} \in \mathfrak{M} \cap \mathfrak{N}$. However, since $\alpha_n \neq 0$ for all n and $\lambda_n \neq 0$ for some $n \geq 1$, the vector ψ is nonzero. This contradicts $\mathfrak{M} \cap \mathfrak{N} = \{0\}$.

PROOF OF THEOREM 1. Let B be an operator on H which is similar to A . That is, there is an invertible operator T on H such that $B = TAT^{-1}$. Assume that B has a nontrivial reducing subspace \mathfrak{M} . Then \mathfrak{M} and its orthogonal complement \mathfrak{M}^\perp are invariant under $B^* = (T^*)^{-1}A^*T^*$. Hence $T^*\mathfrak{M}$ and $T^*\mathfrak{M}^\perp$ are nontrivial invariant subspaces of A^* , and further H is decomposed in the direct sum

$$H = T^*\mathfrak{M} + T^*\mathfrak{M}^\perp,$$

which is impossible by the above lemma.

2. Before proving Theorem 2, let us recall that a spectral operator is the sum $S+N$ of a scalar type operator S and a quasi-nilpotent operator N commuting with S , and a scalar type operator on a Hilbert space is similar to a normal operator (see [1]). The proof is completed by the same argument as that employed in [4].

PROOF OF THEOREM 2. We need only to prove that if the weighted shift A is spectral, then it is quasi-nilpotent. In this case, as we have mentioned above, A is similar to a spectral operator \tilde{A} whose scalar part \tilde{S} is normal. By Theorem 1, \tilde{A} is irreducible, that is to say, the von Neumann algebra generated by \tilde{A} is the algebra $\mathfrak{L}(H)$ of all operators on H . Since \tilde{S} commutes with \tilde{A} , the normality of \tilde{S} implies that \tilde{S}^* also commutes with \tilde{A} . Thus \tilde{S} belongs to the center of $\mathfrak{L}(H)$ and hence \tilde{S} must be a scalar multiple λI of the identity operator I . This implies that $A = \lambda I + N$, where N is quasi-nilpotent. It follows that the spectrum $\sigma(A)$ of A consists of only one point λ . But $\sigma(A)$ is circular symmetry, i.e., if $|\alpha| = 1$, then $\alpha\lambda \in \sigma(A)$ (see [2: Problem 75]). Thus λ must be 0.

ADDED IN PROOF. H. Behncke has proved independently Theorem 1 in his manuscript, *A class of irreducible operators*, but his proof is entirely different from ours.

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