A NOTE ON SEMIFREE ACTIONS OF S^1 ON HOMOTOPY SPHERES

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1. **Introduction.** Let $(S^1, \Sigma^{n+2k}, \Sigma^n)$, $k \geqq 2$, denote a semifree differentiable action of S^1 on homotopy $(n+2k)$-sphere Σ^{n+2k} with the fixed point set the homotopy n-sphere Σ^n, that is, S^1 acts freely outside Σ^n. We shall call Σ^n *untwisted* if the normal bundle of Σ^n in Σ^{n+2k} is trivial [2]. The purpose of this note is to study the semifree differentiable actions of S^1 on homotopy spheres with homotopy spheres as untwisted fixed point sets. In fact, we shall establish the following theorems by using the results of R. Lee [7].

THEOREM 1. *Let* $n \equiv 3 \pmod 4$ *and* $4k-1 \leqq n$. If *the homotopy sphere* Σ^{n+2k} *admits a semifree differentiable* S^1 *action with untwisted fixed point set* Σ^n, *then* Σ^{n+2k} *admits infinitely many differentiably distinct, semifree, differentiable* S^1 *actions with untwisted fixed point set* Σ^n.

THEOREM 2. *There are an infinite number of distinct semifree* S^1 *actions on* S^{17} *with every element of* $32\theta_{11}$ *as untwisted fixed point set. For notation* θ_n, *see* [4, p. 504].

2. **Proofs of the theorems.**

DEFINITION 1. The *standard* (*semifree*) *action* of S^1 on S^{n+2k} with *untwisted fixed point set* S^n is defined as follows: Write

$$S^{n+2k} = \left\{ (x_1, \cdots, x_{n+1}, z_1, \cdots, z_k) \in R^{n+1} \times C^k \,\middle|\, \sum_{i=1}^{n+1} x_i^2 + \sum_{j=1}^{k} |z_j|^2 = 1 \right\}$$

For $g \in S^1$, $(x_1, \cdots, x_{n+1}, z_1, \cdots, z_k) \in S^{n+2k}$, the action is defined by

$$g(x_1, \cdots, x_{n+1}, z_1, \cdots, z_k) = (x_1, \cdots, x_{n+1}, gz_1, \cdots, gz_k).$$

We denote this action simply by (S^1, S^{n+2k}, S^n).

Let $(S^1, \Sigma^{n+2k}, \Sigma^n)$ be any semifree differentiable action of S^1 on Σ^{n+2k} with untwisted fixed point set Σ^n. Then the action around the fixed point set is equivalent to $(S^1, \Sigma^n \times D^{2k}, \Sigma^n \times 0)$ given by

$$g(x, z_1, \cdots, z_k) = (x, gz_1, \cdots, gz_k),$$

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for all $g \in S^1$, $x \in \Sigma^n$ and $(z_1, \cdots, z_k) \in D^{2k}$, where gz_i $(i=1, \cdots, k)$ is the complex multiplication of g and z_i in C. Let $-\Sigma^n$ be the homotopy sphere Σ^n with orientation reversed and $(S^1, -\Sigma^{n+2k}, -\Sigma^n)$ be the action induced by $(S^1, \Sigma^{n+2k}, \Sigma^n)$. Let $(S^1, \Sigma_1^{n+2k}, \Sigma_1^n)$ and $(S^1, \Sigma_2^{n+2k}, \Sigma_2^n)$ be any two semifree actions with untwisted fixed point sets. Since the actions around the fixed point sets Σ_1^n and Σ_2^n are equivalent, the equivarient connected sum $(S^1, \Sigma_1^{n+2k} \# \Sigma_2^{n+2k}, \Sigma_1^n \# \Sigma_2^n)$ is well defined. Two semifree S^1 actions with untwisted fixed point sets, $(S^1, \Sigma_1^{n+2k}, \Sigma_1^n)$ and $(S^1, \Sigma_2^{n+2k}, \Sigma_2^n)$ are said to be *equivalent* if the underlying knots $(\Sigma_1^{n+2k}, \Sigma_1^n)$ and $(\Sigma_2^{n+2k}, \Sigma_2^n)$ are isotopic. The equivalence class of $(S^1, \Sigma^{n+2k}, \Sigma^n)$ is denoted by $[S^1, \Sigma^{n+2k}, \Sigma^n]$. Let $SF(n+2k, n)$ be the set of equivalent classes. Then $SF(n+2k, n)$ is an abelian group under the equivariant connected sum operation:

$$[S^1, \Sigma_1^{n+2k}, \Sigma_1^n] + [S^1, \Sigma_2^{n+2k}, \Sigma_2^n] = [S^1, \Sigma_1^{n+2k} \# \Sigma_2^{n+2k}, \Sigma_1^n \# \Sigma_2^n].$$

For if $(S^1, \Sigma^{n+2k}, \Sigma^n)$ is a representative of $[S^1, \Sigma^{n+2k}, \Sigma^n]$ in $SF(n+2k, n)$, then the imbedding $\Sigma^n \# -\Sigma^n \to \Sigma^{n+2k} \# -\Sigma^{n+2k}$ is isotopic to the standard imbedding $S^n \to S^{n+2k}$. Hence $SF(n+2k, n)$ is an abelian group with zero element $[S^1, S^{n+2k}, S^n]$.

Let $SF(n+2k, n)^*$ and $SF(n+2k, n)^{**}$ be the subgroups of $SF(n+2k, n)$ consisting of elements $[S^1, \Sigma^{n+2k}, \Sigma^n]$ with $\Sigma^{n+2k} = S^{n+2k}$ and $\Sigma^n = S^n$ respectively. Define

$$SF(n + 2k, n)^{\sim} = SF(n + 2k, n)^* \cap SF(n + 2k, n)^{**}.$$

Let us recall that $\theta^{n+2k,n}$ denotes the group of isotopy classes of knotted n-spheres in S^{n+2k} [**6**], and bP_n be as in [**4**, p. 510].

DEFINITION 2. We define the homomorphisms

$$\alpha(n + 2k, n): SF(n + 2k, n) \to \theta_{n+2k}$$

and

$$\beta(n + 2k, n): SF(n + 2k, n) \to \theta_n$$

by

$$\alpha(n + 2k, n)[S^1, \Sigma^{n+2k}, \Sigma^n] = \Sigma^{n+2k}$$

and

$$\beta(n + 2k, n)[S^1, \Sigma^{n+2k}, \Sigma^n] = \Sigma^n.$$

The diagram below is clearly commutative with exact rows and columns:

$$\begin{array}{ccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \rightarrow & SF(n+2k,n)^{\sim} & \rightarrow & SF(n+2k,n)^{*} & & \\
 & & \downarrow & & \downarrow & & \\
0 & \rightarrow & SF(n+2k,n)^{**} & \rightarrow & SF(n+2k,n) & \xrightarrow{\beta(n+2k,n)} & \theta_n \\
 & & & & \downarrow{\scriptstyle \alpha(n+2k,n)} & & \\
 & & & & \theta_{n+2k} & &
\end{array}$$

LEMMA 1. *The groups* $SF(n+2k,n)^{\sim}$, $SF(n+2k,n)^{*}$, $SF(n+2k,n)^{**}$ *and* $SF(n+2k,n)$ *are infinite if* $n\equiv 3 \pmod 4$ *and* $4k-1\leqq n$.

PROOF. Let $\Sigma^{n+2k,n}$ be the kernel of $\theta^{n+2k,n}\rightarrow\theta_n$, and let $\Sigma_0^{n+2k,n}$ be the subgroup of $\Sigma^{n+2k,n}$ of knotted spheres which bound framed submanifolds in S^{n+2k}. Then $\Sigma_0^{n+2k}\approx Z$ under the hypotheses by [6]. According to [7, 5.2, 5.4], there exist infinitely many elements in $\Sigma_0^{n+2k,n}$ such that every element $[S^{n+2k}, S^n]$ has a representative (S^{n+2k}, S^n) which can be realized as the fixed point knot of a semifree differentiable S^1 action on S^{n+2k} with untwisted fixed point set S^n. Thus $SF(n+2k,n)^{\sim}$ is infinite. This proves Lemma 1.

To prove Theorem 1, let $\Sigma^{n+2k}\in\operatorname{Im}\alpha(n+2k,n)$ and $\beta\in\operatorname{Im}\beta(n+2k,n)$. Then $\alpha(n+2k,n)^{-1}(\Sigma^{n+2k})\cap\beta(n+2k,n)^{-1}(\Sigma^n)$ contains infinitely many elements by Lemma 1. The proof is complete.

Now we recall a theorem of Browder [2, (6.2)]:

THEOREM 3 (BROWDER). *Suppose* $n\equiv 3 \pmod 4$ *and* $k>1$, k *odd, and let* $I_0(CP^{k-1}\times S^n)=\{\Sigma\in\theta_{n+2k-2}\mid (CP^{k-1}\times S^n)\#\Sigma$ *is diffeomorphic to* $CP^{k-1}\times S^n\}\cap bP_{n+2k-1}$ *and* $l=$ *order of* $I_0(CP^{k-1}\times S^n)$. *Then an element* $\Sigma^n\in bP_{n+1}$, $n>3$, *occurs as an untwisted fixed point set of a semifree* S^1 *action on a homotopy sphere* Σ^{n+2k} *if and only if* $\Sigma^n\in(m_{n,k}/l)bP_n$, *where* $m_{n,k}$ *is the order of* bP_{n+2k-1}.

Applying Theorem 3 to $n=11$ and $k=3$, since the orders of θ_{17} and bP_{16} are 16 and 8128 respectively, we may use the connected sum method to show that there are semifree S^1 actions on S^{17} with every element of $8128\theta_{11}=32\theta_{11}=Z_{31}$ as untwisted fixed point set. But $4k-1=n$, so we can apply Theorem 1. This completes the proof of Theorem 2.

Browder has found some exotic spheres in $\operatorname{Im}\beta(n+2k,n)$ [2]. In general the groups $\operatorname{Im}\alpha(n+2k,n)$ and $\operatorname{Im}\beta(n+2k,n)$ are hard to compute.

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