A REMARK ON THE DENSITY CHARACTER OF FUNCTION SPACES

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Concerning the density character of function spaces, the most general theorem we know is the unique theorem in [4]—whose brilliant proof carries over verbatim to arbitrary infinite cardinals—and its generalization in [5, (J) and (D)]. The present note investigates the case of the range space separable without countable base as in Michael's theorem [4]. A positive result is given for the range a convex subset of a locally convex space, or an injective uniform space.

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Notations. We shall denote by

\( C(X, Y) \) the set of all continuous maps \( X \to Y \).

\( C_c(X, Y) \) the set \( C(X, Y) \) with the compact-open topology.

\( C_p(X, Y) \) the set \( C(X, Y) \) with the product topology.

\( w(X) \) the weight of \( X \) = least cardinal of a basis for the topology of \( X \).

\( dc(X) \) the density character of \( X \) = least cardinal of a dense subset of \( X \).

**Theorem.** Let \( X \) be a topological space and \( Y \) a convex subset of a locally convex space \( Z \). If \( Y \) is equipped with the induced topology, then

\[
dc(C_c(X, Y)) \leq w(X) \cdot dc(Y)
\]

provided that \( w(X) \) or \( dc(Y) \) is infinite.

**Proof.** Because a translation is a homeomorphism preserving convexity, we may assume—as we do—that \( 0 \in Y \). Let \( I^n \) be the product of the unit interval \( n \) times \( (n \in \mathbb{Z}^+) \), and let \( J_n \) be the subset of \( I^n \) defined by the condition: \( x \in J_n \) iff the sum of the coordinates of \( x \) is less than or equal to 1. Because \( Y \) is convex and \( 0 \in Y \), there is a natural map \( \Phi_n: C_c(X, J_n) \times Y^n \to C_c(X, Y) \) defined by

\[
\Phi_n(f, y) = \left( x \mapsto \sum_{i=1}^n f_i(x)y_i \right)
\]

where \( f = (f_1, \ldots, f_n) \in C(X, J_n) \) and \( y = (y_1, \ldots, y_n) \in Y^n \). Clearly \( \Phi_n \) is continuous. From the fact that compact \( T_3 \)-spaces (as all compact subsets of \( X \) are under the weak topology determined by an \( f \in C(X, Y) \)) admit partitions of unity and from the fact that \( Y \) is \( AE \) (metric)—hence \( AE \) (pseudometric)—by Dugundji extension theo-
rem [2, ix.6.1], it is easily seen that \( \bigcup_{n=1}^{\aleph_0} \Phi_n(C_c(X, J_n) \times Y^n) \) is dense in \( C_c(X, Y) \). By the same proof of the unique theorem in [4] we may establish the following theorem:

*If \( \aleph \) is an infinite cardinal and \( X, Y \) are arbitrary topological spaces with \( w(X), w(Y)^{\not\in \aleph} \), then hereditarily \( dc(C_c(X, Y))^{\leq \aleph} \).

This theorem implies that \( dc(C_c(X, J_n))^{\leq w(X)} \cdot N_0 \) because \( J_n \) is separable metric. Moreover, \( dc(Y^n)^{\leq dc(Y) \cdot N_0} \). Therefore, by the continuity of \( \Phi_n \),

\[
dc(\Phi_n(C_c(X, J_n) \times Y^n))^{\leq w(X) \cdot dc(Y) \cdot N_0^2} = w(X) \cdot dc(Y).
\]

Hence the result. Q.E.D.

The above result is poor in the case of \( X \) discrete: If \( Card(X) = 2^{\aleph_0} \), then \( C_c(X, R) \) is separable by [2, viii.7.2.(3)], although the above theorem only gives a dense subset of cardinality \( \leq 2^{\aleph_0} \). But the above theorem is best possible in case \( X \) is compact: If \( X \) is the product of the unit interval \( 2^{\aleph_0} \) times, then \( X \) is compact, separable (by [2, viii.7.2.(3)]) and nonmetrizable; therefore \( w(X) = 2^{\aleph_0} \), and \( C_c(X, R) \) cannot be separable, because otherwise \( X \) would be metrizable by [1, §3, Theorem 1].

The property of the above theorem is not hereditary: Let \( K \) be the Cantor space. Because \( K \) is uncountable, \( C_0(K, C_0([0, 2])) \) is not hereditarily separable as the proof of [5, Lemma 10.6] indicates. And \( C_0(K, [0, 2])) \) is a separable convex subset of a locally convex space — the product \( R^K \).

A natural question: Is the above theorem true for general \( Y \)? A positive answer can be derived as a corollary of the above theorem via a uniform embedding into a product of Banach spaces when \( Y \) is an injective uniform space for some uniformity (according to [3, p. 39], a uniform space \( Y \) is called injective *iff*, whenever \( A \) is a uniform subspace of \( X \), every uniformly continuous map \( A \rightarrow Y \) can be extended to a uniformly continuous map \( X \rightarrow Y \).

**References**


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