A REMARK ON THE DENSITY CHARACTER OF FUNCTION SPACES

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Concerning the density character of function spaces, the most general theorem we know is the unique theorem in [4]—whose brilliant proof carries over verbatim to arbitrary infinite cardinals—and its generalization in [5, (J) and (D)]. The present note investigates the case of the range space separable without countable base as in Michael's theorem [4]. A positive result is given for the range a convex subset of a locally convex space, or an injective uniform space.

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Notations. We shall denote by

$C(X, Y)$ the set of all continuous maps $X \to Y$.

$C_c(X, Y)$ the set $C(X, Y)$ with the compact-open topology.

$C_p(X, Y)$ the set $C(X, Y)$ with the product topology.

$w(X)$ the weight of X = least cardinal of a basis for the topology of X.

$\mathrm{dc}(X)$ the density character of X = least cardinal of a dense subset of X.

THEOREM. *Let X be a topological space and Y a convex subset of a locally convex space Z. If Y is equipped with the induced topology, then*

$$\mathrm{dc}(C_c(X, Y)) \leqq w(X) \cdot \mathrm{dc}(Y)$$

provided that $w(X)$ *or* $\mathrm{dc}(Y)$ *is infinite.*

PROOF. Because a translation is a homeomorphism preserving convexity, we may assume—as we do—that $0 \in Y$. Let I^n be the product of the unit interval n times ($n \in \boldsymbol{Z}^+$), and let J_n be the subset of I^n defined by the condition: $x \in J_n$ *iff* the sum of the coordinates of x is less than or equal to 1. Because Y is convex and $0 \in Y$, there is a natural map Φ_n: $C_c(X, J_n) \times Y^n \to C_c(X, Y)$ defined by

$$\Phi_n(f, y) = \left(x \mapsto \sum_{i=1}^{n} f_i(x) y_i\right)$$

where $f = (f_1, \cdots, f_n) \in C(X, J_n)$ and $y = (y_1, \cdots, y_n) \in Y^n$. Clearly Φ_n is continuous. From the fact that compact T_3-spaces (as all compact subsets of X are under the weak topology determined by an $f \in C(X, Y)$) admit partitions of unity and from the fact that Y is AE(metric)—hence AE(pseudometric)—by Dugundji extension theo-

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rem [2, ix.6.1], it is easily seen that $\bigcup_{n=1}^{\infty} \Phi_n(C_c(X, J_n) \times Y^n)$ is dense in $C_c(X, Y)$. By the same proof of the unique theorem in [4] we may establish the following theorem:

If $\aleph$ *is an infinite cardinal and* X, Y *are arbitrary topological spaces with* $w(X)$, $w(Y) \leqq \aleph$, *then hereditarily* $\mathrm{dc}(C_c(X, Y)) \leqq \aleph$.

This theorem implies that $\mathrm{dc}(C_c(X, J_n)) \leqq w(X) \cdot \aleph_0$ because J_n is separable metric. Moreover, $\mathrm{dc}(Y^n) \leqq \mathrm{dc}(Y) \cdot \aleph_0$. Therefore, by the continuity of Φ_n,

$$\mathrm{dc}(\Phi_n(C_c(X, J_n) \times Y^n)) \leqq w(X) \cdot \mathrm{dc}(Y) \cdot \aleph_0^2 = w(X) \cdot \mathrm{dc}(Y).$$

Hence the result. Q.E.D.

The above result is poor in the case of X discrete: If $\mathrm{Card}(X) = 2^{\aleph_0}$, then $C_c(X, \boldsymbol{R})$ is separable by [2, viii.7.2.(3)], although the above theorem only gives a dense subset of cardinality $\leqq 2^{\aleph_0}$. But the above theorem is best possible in case X is compact: If X is the product of the unit interval $2^{\aleph_0}$ times, then X is compact, separable (by [2, viii.7.2.(3)]) and nonmetrizable; therefore $w(X) = 2^{\aleph_0}$, and $C_c(X, \boldsymbol{R})$ cannot be separable, because otherwise X would be metrizable by [1, §3, Theorem 1].

The property of the above theorem is not hereditary: Let $\boldsymbol{K}$ be the Cantor space. Because $\boldsymbol{K}$ is uncountable, $C_p(\boldsymbol{K}, C_p(\boldsymbol{K}, [0, 2]))$ is not hereditarily separable as the proof of [5, Lemma 10.6] indicates. And $C_p(\boldsymbol{K}, [0, 2]))$ is a separable convex subset of a locally convex space —the product $\boldsymbol{R}^{\boldsymbol{K}}$.

A natural question: Is the above theorem true for general Y? A positive answer can be derived as a corollary of the above theorem via a uniform embedding into a product of Banach spaces when Y is an injective uniform space for some uniformity (according to [3, p. 39], a uniform space Y is called injective *iff*, whenever A is a uniform subspace of X, every uniformly continuous map $A \rightarrow Y$ can be extended to a uniformly continuous map $X \rightarrow Y$).

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