NOTE ON RELATIVE p-BASES OF PURELY INSEPARABLE EXTENSIONS

J. N. MORDESON AND B. VINOGRADE

Throughout this note L/K denotes a purely inseparable field extension of characteristic p and nonzero exponent. In [**5**, p. 745], Rygg proves that when L/K has bounded exponent, then a subset M of L is a relative p-base of L/K if and only if M is a minimal generating set of L/K. The purpose of this note is to answer the following question: If every relative p-base of L/K is a minimal generating set, then must L/K be of bounded exponent? The answer is known to be yes when K and L^{p^i} are linearly disjoint, $i=1, 2, \cdots$, see [**1**]. We give two examples for which the answer is no: One in which the maximal perfect subfield of L is contained in K, and the other in which it is not.

The following lemmas are needed for our examples. An intermediate field L' of L/K is called proper if $K \subseteq L' \subset L$.

LEMMA 1. *Every relative p-base of L/K ([**2**, p. 180]) is a minimal generating set of L/K if and only if there does not exist a proper intermediate field L' of L/K such that $L=L'(L^p)$.*

PROOF. If $L=L'(L^p)$, where L' is a proper intermediate field of L/K, then L' contains a relative p-base M of L/K. Thus $L \supset L' \supseteq K(M)$. Conversely, if there exists a relative p-base M of L/K such that $L \supset K(M)$, then $L=L'(L^p)$, where $L'=K(M)$. Q.E.D.

LEMMA 2. *Suppose $L=K(m_1, m_2, \cdots)$, where $m_i \in K(m_{i+1})$, $i=1, 2, \cdots$. Then K, $K(m_i^{p^{j_i}})$, L are the intermediate fields of L/K, $0 \leqq j_i < e_i$ (e_i the exponent of m_i over $K(m_{i-1})$), $i=1, 2, \cdots$, where $K(m_0)$ means K.*

PROOF. Let e_i' denote the exponent of m_i over K, $i=1, 2, \cdots$. By [**2**, p. 196, Exercise 5], the intermediate fields of $K(m_s)/K$ are $K(m_s^{p^{j_s}})$, $0 \leqq j_s \leqq e_s'$. If $0<t<s$, then $K(m_t) \subset K(m_s)$, whence $K(m_t) = K(m_s^{p^{e_s'-e_t'}})$. Thus the intermediate fields of $K(m_s)/K$ are K, $K(m_i^{p^{j_i}})$, $0 \leqq j_i < e_i$, $i=1, \cdots, s$. Let K' be any intermediate field of L/K. If $[K':K] < \infty$, then K'/K is finitely generated. Hence $K' \subseteq K(m_s)$ for some m_s since $L=\bigcup_{i=1}^{\infty} K(m_i)$. Thus $K'=K(m_i^{p^{j_i}})$ for some m_i by the preceding argument. If $[K':K]=\infty$, then K' is the union over c of $K(c)$ for all $c \in K'$. Now $K(c)=K(m_{i_c}^{p^{j_{i_c}}}) \supset K(m_{i_c-1})$ for some m_{i_c}

Received by the editors January 10, 1969.

and for all $c \in K' - K$ by the previous argument. Since $[K':K] = \infty$, i_c is an unbounded function of c. Thus $K' = L$. Q.E.D.

EXAMPLE 1. L/K is of unbounded exponent, the maximal perfect subfield of L is not contained in K, and every relative p-base of L/K is a minimal generating set of L/K: Let P be a perfect field and $z, y, x_1, x_2, \cdots$ independent indeterminates over P. Let $K = P(z, y, x_1, x_2, \cdots)$ and $L = K(m_1, m_2, \cdots)$, where $m_i = z^{p^{-i-1}} x_i^{p^{-1}} + y^{p^{-1}}$, $i = 1, 2, \cdots$. Clearly, L/K is of unbounded exponent. $P(z^{p^{-1}}, z^{p^{-2}}, \cdots)$ is the maximal perfect subfield of L and is not in K. By Lemma 1, every relative p-base of L/K is a minimal generating set of L/K if we show that $L \neq L'(L^p)$ for any proper intermediate field L' of L/K. We postpone this proof.

EXAMPLE 2. L/K is of unbounded exponent, the maximal perfect subfield of L is contained in K, and every relative p-base of L/K is a minimal generating set of L/K: Let P be a perfect field and $y, x_1, x_2, \cdots$ independent indeterminates over P. Let $K = P(y, x_1, x_2, \cdots)$ and $L = K(m_1, m_2, \cdots)$, where $m_1 = x_1^{p^{-2}}$ and $m_{i+1} = (m_i^p y + x_{i+1})^{p^{-2}}$, $i = 1, 2, \cdots$. Clearly, L/K is of unbounded exponent. It follows that $L = P(y, m_1, m_2, \cdots)$ and that $\{y, m_1, m_2, \cdots\}$ is an algebraically independent set over P. That is, L/P is a pure transcendental extension. Thus P, $P \subseteq K$, is the maximal perfect subfield of L by Corollary 2 of [3, p. 388]. By Lemma 1, it remains to be shown that $L \neq L'(L^p)$ for any proper intermediate field L' of L/K.

We prove simultaneously for Examples 1 and 2 that such a field L' cannot exist. In both examples, it follows that $K(L^p) = K(m_1^p, m_2^p, \cdots)$, $m_i^p \in K(m_{i+1}^p)$ and m_{i+1}^p has exponent 1 over $K(m_i^p)$ for $i = 1, 2, \cdots$. Hence, by Lemma 2, the intermediate fields of $K(L^p)/K$ appear in a chain. Now suppose there exists a proper intermediate field L' of L/K such that $L = L'(L^p)$. Since $L' \not\supseteq K(L^p)$, $L' \cap K(L^p) = K(m_s^p)$ for some integer $s \geqq 0$. We show L' and $K(L^p)$ are linearly disjoint over $K(m_s^p)$ by showing that for every proper intermediate field K' of $K(L^p)/K(m_s{}^p)$, L' and K' are linearly disjoint over $K(m_s^p)$. By Lemma 2, $K' = K(m_t^p)$ for $t \geqq s$. Now m_t^p has exponent $t-s$ over $K(m_s^p) \subseteq L'$. If $((m_t^p)^{p^{t-s}})^{p^{-1}} \in L'$, then we contradict $L' \cap K(L^p) = K(m_s^p)$. Hence the irreducible polynomial of m_t^p over $K(m_s^p)$ remains irreducible over L'. Thus L' and $K(m_t^p)$ are linearly disjoint over $K(m_s^p)$, whence L' and $K(L^p)$ are linearly disjoint.

Since $L = L'(L^p)$, $m_{s+1} \in L'(L^p)$. Hence $m_{s+1} = \sum_{j=0}^{p^{t-s}-1} c_j' (m_t^p)^j$ for some integer t, where $c_j' \in L'$. Now $t \geqq s+2$ since m_{s+1} has exponent 2 over $K(m_s^p)$ and L' has exponent 1 over $K(m_s^p)$. Thus

$$m_{s+1}^p = \sum_j c_j'^p (m_t^p)^{jp}.$$

By the division algorithm, $jp = p^{t-s}q_j + r_j$, $0 \leqq r_j < p^{t-s}$. Hence

$$(*) \qquad m_{s+1}^p = \sum_j \left(c_j'^p (m_t^p)^{p^{t-s}q_j}\right)(m_t^p)^{r_j}$$

and

$$c_j'^p (m_t^p)^{p^{t-s}q_j} = c_j^p \in L' \cap L^p.$$

Writing m_{s+1}^p in terms of m_t^p, we get for Example 1:

$$m_{s+1}^p = (m_t^p)^{p^{t-s-1}} k_0^p x_{s+1} - k_1,$$

where

$$k_0^p = x_t^{-p^{t-s-1}}$$

and

$$k_1 = y^{p^{t-s-1}} x_t^{-p^{t-s-1}} x_{s+1} - y.$$

By (*),

$$(m_t^p)^{p^{t-s-1}} x_{s+1} = k_0^{-p} k_1 + k_0^{-p} \sum_j c_j^p (m_t^p)^{r_j}.$$

Hence, by the linear disjointness of L' and $K(L^p)$ over $K(m_s^p)$ and since $\{(m_t^p)^j \mid j = 0, \cdots, p^{t-s}-1\}$ is linearly independent over $K(m_s^p)$, $x_{s+1} \in L' \cap L^p$. Thus $x_{s+1}^{p^{-1}} \in L$, a contradiction.

For Example 2, we get

$$m_{s+1}^p = (m_{s+2}^p)^p y^{p^{-1}} - x_{s+2} y^{-1} = \cdots = (m_t^p)^{p^{t-s-1}} k_0^p y^{p^{-1}} - k_1$$

for suitable $k_0, k_1 \in K$. By an argument similar to that in Example 1, we obtain $y^{p^{-1}} \in L$, a contradiction.

REMARK. Let P denote the maximal perfect subfield of L and M a relative p-base of L/K. Consider the properties: (1) $P \subseteq K$, (2) $P \not\subseteq K$, (3) there exists an M such that $L = K(M)$, and (4) for all M, $L = K(M)$. None implies the other except for (4) implies (3).

For instance, Example 1 shows that (4)$\nrightarrow$(1) and Example 2 shows that (4)$\nrightarrow$(2). Example 2 of [4, p. 333] shows that (3)$\nrightarrow$(4). Letting L be perfect gives us an example showing that (2)$\nrightarrow$(3). We show (1)$\nrightarrow$(3) by giving an example constructed by E. A. Hamann. Let $K = Q(x_1, x_2, \cdots)$ and

$$L = K\left(x_1^{p^{-1}}, (x_1 + x_2^p)^{p^{-2}}, (x_1 + x_2^p + x_1^{p^2})^{p^{-3}}, \cdots\right),$$

where Q is a perfect field and $x_1, x_2, \cdots$ are independent indeterminates over Q. ($L = K(L^p)$ and L/Q is pure transcendental.) The remaining implications are trivial.

References

1. G. Haddix, J. Mordeson and B. Vinograde, *On purely inseparable extensions of unbounded exponent*, Canad. J. Math. (to appear).

2. N. Jacobson, *Lectures in abstract algebra*. Vol. III, Van Nostrand, Princeton, N. J., 1964.

3. S. MacLane, *Modular fields*, Duke Math. J. **5** (1939), 372–393.

4. J. Mordeson and B. Vinograde, *Generators and tensor factors of purely inseparable fields*, Math. Z. **107** (1968), 326–334.

5. P. Rygg, *On minimal sets of generators of purely inseparable field extensions*, Proc. Amer. Math. Soc. **14** (1963), 742–745.

Creighton University and
Iowa State University