# MULTIPLICATIVE INVERSES OF SOLUTIONS FOR VOLTERRA-STIELTJES INTEGRAL EQUATIONS

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H. S. Wall [5] found that, if $n$ is a positive integer, then there is a reversible function from the class $\phi_n$ of $n\times n$ matrices $F$ of complex functions defined on the real line $S$, continuous and of bounded variation on each interval, such that $F(0)=0$ to a class $H_n$ which is established by the condition that

$$W(x, z) = 1 + \int_x^z dF \cdot W(I, z) \tag{1}$$

for each $\{x, z\}$ in $S\times S$. Furthermore, for each number triple $\{x, y, z\}$,

$$W(x, y)W(y, z) = W(x, z) \quad \text{and} \quad W(x, x) = 1. \tag{2}$$

T. H. Hildebrandt [1] dropped the condition of continuity and studied the same type integral equation using the Young integral. He found it necessary to require the existence of certain multiplicative inverses in order to get the existence of such a solution $W$. (See also [4, p. 323].)

J. S. MacNerney [3] extended the one-to-one correspondence to a more general ring setting. In this setting, $S$ is a nondegenerate, linearly ordered set, $R$ is a normed ring with unity, and each of $OA$ and $OM$ is a class of functions from $S\times S$ to $R$. In addition to a correspondence similar to (1) relating a member $W$ in $OM$ with a member $V$ in $OA$, MacNerney uses the continuously continued sum and the continuously continued product: for each $\{a, b\}$ in $S\times S$

$$V(a, b) = {}_a\sum^b [W - 1] \quad \text{and} \quad W(a, b) = {}_a\prod^b [1 + V].$$

The function $W$ is no longer fully multiplicative but, rather, relation (2) holds for all $\{x, y, z\}$ such that $x\leqq y\leqq z$ or $x\geqq y\geqq z$. (See also [2, p. 329].)

In case $W$ is fully multiplicative, it is clear that, for each $\{x, y\}$ in $S\times S$, $W(x, y)$ has a multiplicative inverse—namely $W(y, x)$. In the more general setting developed by MacNerney, but with $S$ the set of real numbers, we shall find a necessary and sufficient condition that, for each $\{x, y\}$ in $S\times S$, $W(x, y)$ shall have a multiplicative inverse

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in $R$. In case $W$ has an inverse, we shall determine a function $g$ having the property that if $M$ is in $OM$ so that

$$M(x, y) = 1 + \int_x^y dg \cdot M(I, y) \qquad \text{for each } \{x, y\} \text{ in } S \times S$$

then $M(x, y) \cdot W(y, x) = 1$. Thus $g$ generates the multiplicative inverse for $W$.

1. Let us suppose that $S$ is the set of real numbers and that $R$ is a ring with unity on which there is defined a real-valued norm with respect to which $R$ is complete.

A function $B$ is said to be order-additive from $S \times S$ into any ring provided that, if $\{x, z\}$ is in $S \times S$ and $x \leqq y \leqq z$ or $x \geqq y \geqq z$, then $B(x, y) + B(y, z) = B(x, z)$ and is said to be order-multiplicative provided that, if $\{x, z\}$ is in $S \times S$ and $x \leqq y \leqq z$ or $x \geqq y \geqq z$, then $B(x, y)B(y, z) = B(x, z)$. As in [3], we shall let $OA^+$ denote the class of all order-additive functions from $S \times S$ to the set of nonnegative real numbers, and let $OM^+$ denote the class of all order-multiplicative functions from $S \times S$ to the set of real numbers not less than one. The class $OA$ shall consist of all order-additive functions $V$ from $S \times S$ to $R$ for which it is true that there is a function $\alpha$ in $OA^+$ with the property that $|V(a, b)| \leqq \alpha(a, b)$ for each $\{a, b\}$, and $OM$ shall consist of all order-multiplicative functions $W$ from $S \times S$ to $R$ for which there exists a function $\mu$ in $OM^+$ with the property that $|W(a, b) - 1| \leqq \mu(a, b) - 1$ for each $\{a, b\}$. Let $OB$ denote the class of functions $F$ from $S$ to $R$ such that $dF$ is in $OA$, where $dF(x, z)$ is defined to be $F(z) - F(x)$. Note that $OB$ is precisely the class of functions from $S$ to $R$ each member of which is of bounded variation on each interval.

In [3, 150], Professor MacNerney defines the continuously continued sum and product. We indicate the definition: If $h$ is a function from $S \times S$ to $R$ and $a$ and $b$ are in $S$, then

$$_a\sum^b h \sim \sum_1^n h(t_{p-1}, t_p) \quad \text{and} \quad _a\prod^b h \sim \prod_1^n h(t_{p-1}, t_p)$$

where $\{t_p\}_0^n$ is a subdivision of $\{a, b\}$.

If $\alpha$ is in $OA^+$ and $a < b$, then $\sum_{a \leqq x < b} \alpha(x, x^+)$ exists and is, again, the limit in the sense of successive refinements of subdivisions. In this case, the limit is the least upper bound for all approximating sums since the "approximations" increase with increasing subdivisions and are bounded by $\alpha(a, b)$. If $V$ is in $OA$, the existence of $\sum_{a \leqq x < b} \alpha(x, x^+)$ gives the existence of $\sum_{a \leqq x < b} V(x, x^+)$. A similar integral exists for the "pairs" $(x^-, x)$, $(x^+, x)$, and $(x, x^-)$.

2. In this section we shall investigate an integral with approximating sums $\sum_t [1+V]^{-1}V$.

LEMMA 1. *Suppose that each of U and V is in OA, $a<b$, and $c>0$. There is a subdivision s of $\{a, b\}$ such that if $\{t_p\}_0^n$ refines s, then*
(3) $\sum_1^n | V(t_{p-1}^+, t_p^-)\cdot U(t_{p-1}^+, t_p^-)| < c$ *and*
(4) $\sum_1^n | V(t_{p-1}, t_{p-1}^+)\cdot U(t_p^-, t_p)| < c$.

REMARK. Thus, if each of $f$ and $g$ is of bounded variation on the interval $[a, b]$ and $c>0$, then there is a subdivision $s$ of $[a, b]$ so that if $t$ is a refinement of $s$, then

$$\sum_1^n | [f(t_{p-1}^+) - f(t_{p-1})]\cdot[g(t_p) - g(t_p^-)]| < c.$$

LEMMA 2. *Suppose that V is in OA, $\{a, b\}$ is in $S\times S$, and for each x in S such that $a\leqq x\leqq b$ or $a\geqq x\geqq b$, each of $1+V(x, x^+)$, $1+V(x^-, x)$, $1+V(x^+, x)$, and $1+V(x, x^-)$ has a multiplicative inverse in R. There is a subdivision s of $\{a, b\}$, a number A, and a number B such that, if $\{t_p\}_0^n$ is a refinement of s, then $[1+V(t_{p-1}, t_p)]^{-1}$ exists and $A<| [1+V(t_{p-1}, t_p)]^{-1}| <B$ for $p=1, 2, \cdots, n$.*

REMARK. If, with the supposition of Lemma 2, $\alpha$ is in $OA^+$ such that $| V| \leqq\alpha$ and $\{t_p\}_0^n$ is a refinement of the subdivision $s$ given in Lemma 2, then, using [**3**, Lemma 2.1],

$$\left| \prod_1^n [1 + V(t_{p-1}, t_p)]^{-1}\right| = \left| \prod_1^n [1 - [1 + V(t_{p-1}, t_p]^{-1}V(t_{p-1}, t_p)]\right|$$
$$\leqq \prod_1^n [1 + B\cdot\alpha(t_{p-1}, t_p)] \leqq \exp(B\cdot\alpha(a, b)).$$

To prove the next theorem, we use the following elementary arithmetic in $R$. If $r$ is in $R$ and $1+r$ has a multiplicative inverse in $R$, then $(1+r)^{-1}r=r(1+r)^{-1}$. Furthermore, if each of $a$, $b$, and $c$ is in $R$ and each of $1+a$, $1+c$, and $1+a+b+c$ has a multiplicative inverse in $R$, then

$$\begin{aligned}(1 + a)^{-1}a + b + (1 + c)^{-1}c - (1 + a + b + c)^{-1}(a + b + c)\\ = b(a + b + c)(1 + a + b + c)^{-1}\\ + (1 + c)^{-1}(1 + a)^{-1}[a + c + ac]b(1 + a + b + c)^{-1}\\ + (1 + c)^{-1}(1 + a)^{-1}ac(1 + a + b + c)^{-1}\\ + (1 + c)^{-1}c(1 + a)^{-1}a.\end{aligned}$$

Consequently, in addition to the inequality in the above remark, we may add that if $a<b$ then

$$\sum_1^n \Big| [1 + V(t_{p-1}, t_{p-1}^+)]^{-1} V(t_{p-1}, t_{p-1}^+) + V(t_{p-1}^+, t_p^-) + [1 + V(t_p^-, t_p)]^{-1} V(t_p^-, t_p) - [1 + V(t_{p-1}, t_p)]^{-1} V(t_{p-1}, t_p) \Big|$$

$$\leqq \sum_1^n \{\alpha(t_{p-1}^+, t_p^-)\alpha(t_{p-1}, t_p)\cdot B + (B^3/4)[\alpha(t_{p-1}, t_{p-1}^+) + \alpha(t_p^-, t_p) + \alpha(t_{p-1}, t_{p-1}^+)\cdot\alpha(t_p^-, t_p)]\alpha(t_{p-1}^+, t_p^-) + ((B^3 + B^2)/4)[\alpha(t_{p-1}, t_{p-1}^+)\alpha(t_p^-, t_p)]\}.$$

A similar inequality holds in case $a>b$. Thus, Lemma 1 and its analogue in case $a>b$ give the following theorem.

THEOREM 1. *If $V$ is in $OA$, $a$ and $b$ are in $S$, and, for each $x$ such that $a\leqq x\leqq b$ or $a\geqq x\geqq b$, each of $1+V(x, x^+)$, $1+V(x^-, x)$, $1+V(x^+, x)$, and $1+V(x, x^-)$ has a multiplicative inverse in $R$, then ${}_a\sum^b [1+V]^{-1}V$ exists and is*

$$V(a, b) + \sum_{a\leq x<b} \{[1 + V(x, x^+)]^{-1}V(x, x^+) - V(x, x^+)\} + \sum_{a<y\leq b} \{[1 + V(y^-, y)]^{-1}V(y^-, y) - V(y^-, y)\}$$

*or*

$$V(a, b) + \sum_{a\geq x>b} \{[1 + V(x, x^-)]^{-1}V(x, x^-) - V(x, x^-)\} + \sum_{a>y\geq b} \{[1 + V(y^+, y)]^{-1}V(y^+, y) - V(y^+, y)\}$$

*according as to whether $a<b$ or $a>b$.*

REMARK. In a similar manner, one can show that if each of $f$ and $g$ is of bounded variation on the interval $[a, b]$ and $1+dg$ has a multiplicative inverse in $R$ at the discontinuities of $g$, then ${}_a\int^b df/(1+dg)$ exists.

DEFINITION. Let $OAI$ denote the class of functions $V$ in $OA$ for which it is true that, for each $x$ in $S$, each of $1+V(x, x^+)$, $1+V(x^-, x)$, $1+V(x^+, x)$, and $1+V(x, x^-)$ has a multiplicative inverse in $R$.

THEOREM 2. *There is a reversible function $G$ from $OAI$ onto $OAI$ which has the following properties: If $V$ is in $OAI$, then*

(i) $G[G[V]]=V$,

(ii) $G[V](a, b) = -V(b, a)$ *for each* $\{a, b\}$ *in* $S\times S$ *if and only if* ${}_a\sum^b |V^2| = 0$ *for each* $\{a, b\}$ *in* $S\times S$,

(iii) ${}_a\prod^b [1+G[V]]\cdot{}_b\prod^a [1+V] = 1$ *for each* $\{a, b\}$ *in* $S\times S$, *and*

(iv) $G[V](a, b) = -{}_b\sum^a [1+V]^{-1}V$ *for each* $\{a, b\}$ *in* $S\times S$.

Indication of Proof. We let $G$ denote the collection of ordered pairs to which $\{V, U\}$ belongs only in case $V$ is in $OAI$ and $U(a, b) = -{}_b\sum^a [1+V]^{-1}V$ and $U(a, a) = 0$ for each $\{a, b\}$ in $S\times S$. We see that if $V$ is in $OAI$ then $G[V]$ is in $OA$. Furthermore, $1+G[V](x, x^+) = [1+V(x^+, x)]^{-1}$, and there is a similar relation for the "pairs" $(x^-, x)$, $(x^+, x)$, and $(x, x^-)$. Hence $G[V]$ is in $OAI$. Thus, it remains to show that the function $G$ established by (iv) of Theorem 2 has properties (i), (ii), and (iii).

The nature of the convergence of the approximating sums for ${}_a\sum^b [1+V]^{-1}V$ is indicated in the following lemma.

Lemma 3. *If* $V$ *is in* $OAI$, $a$ *and* $b$ *are in* $S$, *and* $c > 0$, *then there is a subdivision* $s$ *of* $\{a, b\}$ *such that, if* $\{t_p\}_0^n$ *is a refinement of* $s$, *then*

$$\sum_1^n \left| [1 + V(t_{p-1}, t_p)]^{-1}V(t_{p-1}, t_p) + G[V](t_p, t_{p-1}) \right| < c.$$

To establish part (i) of Theorem 2, suppose that $a$ and $b$ are in $S$, $V$ is in $OAI$, $s$ is a subdivision of $\{a, b\}$, and $B$ is a number having the property that if $\{t_p\}_0^n$ is a refinement of $s$, then each of $[1+V(t_{p-1}, t_p)]$ and $[1+G[V](t_p, t_{p-1})]$ has a multiplicative inverse in $R$ and $|[1+G[V](t_p, t_{p-1})]^{-1}| < B$, for $p = 1, 2, \cdots, n$. Since $G[G[V]](a, b)$ is $-{}_b\sum^a [1+G[V]]^{-1}G[V]$, we consider the following inequality:

$$\begin{aligned}
&\left| -\sum_1^n [1 + G[V](t_p, t_{p-1})]^{-1}G[V](t_p, t_{p-1}) - V(a, b) \right| \\
&\leqq \sum_1^n \left| [1 + G[V](t_p, t_{p-1})]^{-1} - 1 - V(t_{p-1}, t_p) \right| \\
&= \sum_1^n \left| [1 + G[V](t_p, t_{p-1})]^{-1}\{[1 + V(t_{p-1}, t_p)]^{-1} - 1 - G[V](t_p, t_{p-1})\} \right. \\
&\qquad\qquad \left. \cdot [1 + V(t_{p-1}, t_p)] \right| \\
&\leqq B\cdot \sum_1^n \left| [1+V(t_{p-1}, t_p)]^{-1}V(t_{p-1}, t_p) + G[V](t_p, t_{p-1}) \right| \cdot (1+\alpha(a, b)).
\end{aligned}$$

We now apply Lemma 3 to get that $G[G[V]] = V$.

Lemma 4. *If* $V$ *is in* $OA$ *and* $\{a, b\}$ *is in* $S\times S$, *then* ${}_a\sum^b |V^2|$ *is*

$\sum_{a\leq x<y\leq b} | V(x, x^+)^2 + V(y^-, y)^2|$ *or* $\sum_{a\geq x>y\geq b} | V(x, x^-)^2 + V(y^+, y)^2|$ *or zero according as* $a<b$, $a>b$, *or* $a=b$.

To establish part (ii) of Theorem 2, suppose that $V$ is in $OAI$ and that $G[V](b, a) = -V(a, b)$ for each $\{a, b\}$ in $S\times S$. Then $-V(x, x^+) = G[V](x^+, x) = -[1+V(x, x^+)]^{-1}V(x, x^+)$. Thus,

$$0 = [1 + V(x, x^+)]^{-1}V(x, x^+)^2$$

or $V(x, x^+)^2 = 0$. In a similar manner, each of $V(x^-, x)^2$, $V(x^+, x)^2$ and $V(x, x^-)^2$ is 0. That ${}_a\sum^b | V^2| = 0$ for each $\{a, b\}$ in $S\times S$ now follows from Lemma 4. Suppose that $V$ is in $OAI$ and ${}_a\sum^b | V^2| = 0$ for each $\{a, b\}$ in $S\times S$. Let $s$ and $B$ be as in Lemma 2 and $t$ be a refinement of $s$.

$$\left| -V(a, b) + \sum_t [1 + V]^{-1}V \right| = \left| -\sum_t [1 + V]^{-1}V^2 \right| \leqq B\cdot \sum_t | V^2 | .$$

Consequently, $G[V](b, a) = -V(a, b)$ for each $\{a, b\}$ in $S\times S$.

REMARK. A companion theorem to Theorem 2 (ii) is [**3**, Theorem 7.31].

Finally, to establish part (iii) of Theorem 2, suppose that $V$ is in $OAI$, each of $\alpha$ and $\beta$ is in $OA^+$ such that $|V| \leqq \alpha$ and $|G[V]| \leqq \beta$, $\{a, b\}$ is in $S\times S$, $\{t_p\}_0^n$ is a subdivision of $\{a, b\}$ such that $[1+V(t_p, t_{p-1})]$ has a multiplicative inverse, and $B$ is a number such that $|[1+V(t_p, t_{p-1})]^{-1}| < B$ for $p = 1, 2, \cdots, n$. We consider the following inequality:

$$\begin{aligned}
&\left| \prod_1^n [1 + G[V](t_{p-1}, t_p)]\cdot \prod_1^n [1 + V(t_{n+1-p}, t_{n-p})] - 1 \right| \\
&\quad = \left| \left\{ \prod_1^n [1 + G[V](t_{p-1}, t_p)] - \prod_1^n [1 + V(t_p, t_{p-1})]^{-1} \right\} \right. \\
&\qquad\qquad \left. \cdot \prod_1^n [1 + V(t_{n+1-p}, t_{n-p})] \right| \\
&\quad \leqq \left| \sum_{j=1}^n \prod_{p=1}^{j-1} [1 + V(t_p, t_{p-1})]^{-1}\{1 + G[V](t_{j-1}, t_j) - [1 + V(t_j, t_{j-1})]^{-1}\} \right. \\
&\qquad\qquad \left. \cdot \prod_{p=j+1}^n [1 + G[V](t_{p-1}, t_p)] \right| \cdot \exp(\alpha(b, a)) \\
&\quad \leqq \exp((B + 1)\cdot\alpha(b, a) + \beta(a, b))\cdot \sum_1^n | G[V](t_{j-1}, t_j) \\
&\qquad\qquad + [1 + V(t_j, t_{j-1})]^{-1}V(t_j, t_{j-1}) | .
\end{aligned}$$

The assertion of the theorem follows.

LEMMA 5. *If $W$ is in $OM$ and, for each $\{a, b\}$ in $S\times S$, $W(a, b)$ has a multiplicative inverse in $R$, then, for each $x$ in $S$, each of $W(x, x^+)$, $W(x^-, x)$, $W(x^+, x)$, and $W(x, x^-)$ has a multiplicative inverse in $R$.*

INDICATION OF PROOF. Suppose that $x<z$ and, for each $y$ in $[x, z]$, $W(y, z)$ has a multiplicative inverse in $R$. Let $y$ be in $[x, z]$ such that $|W(y, z)-W(x^+, z)|<1/(|W(x, x^+)|\cdot|W(x, z)^{-1}|)$. It follows that $|W(x, x^+)W(y, z)W(x, z)^{-1}-1|<1$. Consequently, $W(x, x^+)$ has a multiplicative inverse in $R$. The remainder of the assertion of the lemma follows in a similar manner.

If $V$ is in $OA$ and $W(a, b)={}_a\prod^b [1+V]$ for each $\{a, b\}$ in $S\times S$, then $W=1+V$ at the pairs $(x, x^+)$, $(x^-, x)$, $(x^+, x)$, and $(x, x^-)$. Consequently, in this paper, the following theorem has been established.

THEOREM 3. *If $V$ is in $OA$ and $W(a, b)={}_a\prod^b [1+V]$ for each $\{a, b\}$ in $S\times S$, then these are equivalent:*

(i) *$W(x, y)$ has a multiplicative inverse for each $\{x, y\}$ in $S\times S$, and*

(ii) *$V$ is in $OAI$.*

We may now use these results, together with the formulas on [**4**, pp. 323, 324], to establish the following theorem.

THEOREM 4. *Suppose that $\{K_1, K_2\}$ is a pair which satisfies Axioms* I, II, *and* III *of* [**4**] *and that $\{W_1, W_2\}$ is a pair in $OM\times OM$ such that*

$$W_1(x, z)=1+K_1[W_1(x, I)](x, z) \quad \text{and} \quad W_2(x, z)=1+K_2[W_2(I, z)](x, z)$$

*for each $\{x, z\}$ in $S\times S$. Each of $W_1(x, z)$ and $W_2(x, z)$ has a multiplicative inverse in $R$ for each $\{x, z\}$ in $S\times S$ if and only if, for each $y$ in $S$, each of the following has a multiplicative inverse in $R$:*

$1+K_1[1_y](y, y^+)$ *and* $1+K_1[0_y](y^-, y)$,
$1+K_1[1_y](y, y^-)$ *and* $1+K_1[0_y](y^+, y)$,
$1+K_2[0_y](y, y^+)$ *and* $1+K_2[1_y](y^-, y)$, *and*
$1+K_2[0_y](y, y^-)$ *and* $1+K_2[1_y](y^+, y)$.

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