T_1-COMPLEMENTS OF T_1 TOPOLOGIES

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1. **Introduction.** Let Λ be the family of all T_1 topologies definable on an arbitrary set. When $\tau_1 \in \Lambda$ and $\tau_2 \in \Lambda$, $\tau_1 < \tau_2$ if every set in τ_1 is in τ_2. Then τ_1 is said to be coarser than τ_2 and τ_2 finer than τ_1. Under this order, Λ is a complete lattice. The greatest element of Λ is the discrete topology, 1, and the least element is the cofinite topology $C = \{U \mid U = \varnothing \text{ or } X - U \text{ is finite}\}$.

Recently several papers have been published dealing with the structure of the lattice Λ. An example [2] has been given to show that Λ is not a complemented lattice, unless X is a finite set. However some fairly extensive classes of T_1 topologies have been shown to have T_1-complements, although these classes do not include the spaces most commonly studied in general topology. It is known [4] that the reals with the usual topology have a T_1-complement. One of the results of this paper shows that if a T_1 space has a countable dense metric subspace, then the space has a T_1-complement. Thus, "the most common spaces" (see [4]) do have T_1-complements. Unfortunately, T_1-complementation is not topologically nice. It is easy to see that there are $\aleph_0$ nonhomeomorphic metric spaces all with the same T_1-complement. We show that no infinite Hausdorff first countable space has a Hausdorff first countable T_1-complement. It follows that no infinite metric space has a metric space T_1-complement. The lattice structure itself is apparently quite complex. An example is given of three mutually T_1-complementary topologies.

Most of the proofs given here are straightforward extensions of ideas found in [3] and [4]. From now on, unless explicitly stated otherwise, all sets are assumed to be infinite.

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2. **Complementation in Λ.** First we show that every T_1 space with a countable dense metric subspace has a T_1-complement. By Theorems 2 and 6 of [3] it will suffice to show that every countable nonempty metric space has a T_1-complement.

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LEMMA 1. *Suppose* $(Z, \tau) = M \cup N$ *such that*
(i) M *and* N *are disjoint and* τ*-open,*
(ii) M *is homeomorphic to the rationals with the usual topology,*
(iii) N *is countable and* $\tau | N$ *is discrete.*
Then τ *has a* T_1*-complement.*

PROOF. Suppose first that $N = \varnothing$. In [**4**], a countable dense subset of the reals is constructed and it is proved that the relative topology on this set has a T_1-complement. Since all countable dense-in-themselves metric spaces are homeomorphic [**1**], it is clear that in this case, τ has a T_1-complement.

Suppose $N \neq \varnothing$, but N is finite. By the preceding paragraph, $\tau | M$ has a T_1-complement $(\tau | M)'$. Let τ' be the topology generated by $(\tau | M)'$ and the cofinite topology C on Z. It is clear that every nonempty τ'-open set intersects M. Thus, if $U \in \tau \cap \tau'$, and $U \neq \varnothing$, then U intersects M and $U \cap M \in \tau | M \cap (\tau | M)'$. Thus $U \cap M$ contains all but a finite number of points of M and hence all but a finite number of points of Z. Since it is obvious that the $\sup\{\tau, \tau'\}$ is discrete, τ' is a T_1-complement for τ.

Lastly, suppose N is countably infinite. For this part of the argument, we assume that the reader is familiar with the notation as in [**4**]. In that paper a subset X of the rationals (homeomorphic to the rationals and hence to M) is obtained as the union of pairwise disjoint sets D, A_0, A_1, and S_n, $n = 1, 2, \cdots$. Then X is given a topology T' generated by singletons in D, and sets B_0, B_1, and C_i, $i = 1, 2, \cdots$. It is proved that if T is the usual topology on X, then T and T' are T_1-complements. In what follows, consider the sets just denoted to be subsets of M and let f be a 1-1 function from M onto N. Let τ' be the topology generated by

(i) $\{x\}$, $x \in D \subset M$,
(ii) the family of all cofinite subsets of

$$B_i \cup f[B_i], \qquad i = 0, 1,$$
$$C_i \cup f[C_i], \qquad i = 1, 2, \cdots.$$

We claim that τ' is a T_1-complement for τ. Since

$$(B_0 \cup f[B_0]) \cup (B_1 \cup f[B_1]) = Z,$$

it is clear that τ' is a T_1 topology. A trivial modification of the argument in [**4**] shows that $\sup\{\tau, \tau'\} = 1$. The lemma will now be verified if it can be shown that $\inf\{\tau, \tau'\}$ is the cofinite topology. Notice first that no nonempty element of τ' is contained in N. This follows since every intersection of a finite number of cofinite subsets of the C_i's will contain almost all elements of D outside the union of a finite number of bounded open intervals. Thus every such intersection will contain infinitely many points of both B_0 and B_1, and $B_0 \cap B_1 = \emptyset$. Suppose now that $U \in \tau \cap \tau'$ and $U \neq \emptyset$. Then $U \cap M \neq \emptyset$ and since it is clear that $\tau' | M = (\tau | M)'$, $U \cap M \in (\tau | M) \cap (\tau | M)'$. Hence, by [**4**], $[B_0 - (U \cap M)]$ and $[B_1 - (U \cap M)]$ are finite. Therefore $U \cap M$ contains points of A_0 and A_1. But if $x \in A_0$, the only subbase elements of τ' that contain x are the cofinite subsets of $B_0 \cup f[B_0]$. Thus U is cofinite in $B_0 \cup f[B_0]$ and similarly U is cofinite in $B_1 \cup f[B_1]$.

THEOREM 1. *If the T_1 space (X, τ) has a countable dense metric subspace, then τ has a T_1-complement.*

PROOF. As stated previously, it suffices to show that every countable nonempty metric space has a T_1-complement. Suppose (Y, σ) is a countable nonempty metric space. Let I be $\{y | y$ is an isolated point of $Y\}$. If Cl $I = Y$, then since $(I, \sigma | I)$ clearly has a T_1-complement, one could use either Theorem 5 or Theorem 6 of [**3**] to conclude that (Y, σ) has a T_1-complement. If Cl $I \neq Y$, then it is easily seen that $(Y - \text{Cl } I)$ is dense-in-itself. Furthermore, $(Y - \text{Cl } I) \cup I$ is an open dense subset of Y. Thus, by Lemma 1 and by [**3**], (Y, σ) has a T_1-complement.

We now show that there are $\aleph_0$ nonhomeomorphic metric spaces all with the same T_1-complement. Let (R, S) be the reals with the usual topology and suppose S' is a T_1-complement for S. Let T be the topology on R gotten by adding sets of the form $[0, \epsilon)$, $\epsilon > 0$ to S. Then T and S' are T_1-complements. This holds since it is clear that $\sup\{T, S'\} = 1$ and, on the other hand, if $U \in T \cap S'$, then $U - \{0\} \in S \cap S'$ and thus U belongs to the cofinite topology on R. The same argument remains valid if we cut the reals at any finite number of points or if we isolate any finite number of points of R that are not isolated with respect to S'.

In Theorem 3 of [**3**] it is shown that a Hausdorff topology on a countable set cannot have a Hausdorff T_1-complement. The following theorem for arbitrary sets is a "generalization" of Theorem 3 of [**3**], but it still leaves this question: Can a Hausdorff topology on an uncountable set have a Hausdorff T_1-complement?

THEOREM 2. *Let (X, T) be a first countable Hausdorff space. If T' is T_1-complement for T, then (X, T') cannot be first countable and Hausdorff.*

PROOF. By Theorem 2 of [**3**] T' must be countably compact on cofinite subsets of X. In a first countable Hausdorff space countably compact sets are closed. Hence if (X, T') were Hausdorff and first countable, then T' would be discrete. This would imply that T is the cofinite topology which contradicts the hypotheses. Hence (X, T) is not Hausdorff and first countable.

Now we extend the work in [4] to give three mutually complementary T_1-topologies on a dense subset of the rationals. Before giving this example it seems only natural to ask the following question: How many mutually complementary T_1-topologies can a set X possess?

Now let X be as in [4]. We wish to construct a subset $Y \subset X$ such that $T|Y$ and $T'|Y$ (see proof of Lemma 1) will still be T_1-complements, and construct a new topology on Y which will be complementary to both $T|Y$ and $T'|Y$. First let $E = X - D$. Consider the sequences $S_{0,k}$ and $S_{1,k}$ (k an integer) as defined in [4]. Let r_k (p_k) be the first term in the sequence $S_{0,k}$ ($S_{1,k}$). Then $k - \frac{1}{2} < p_k < k < r_k < k + \frac{1}{2}$.

For each integer k let $T_{0,k}$ be a sequence in $D \cap (r_k, p_{k+1})$ converging to r_k such that the first term of $T_{0,k}$ is $k + \frac{1}{2}$. For each integer k let $T_{1,k}$ be a sequence in $D \cap (p_k, r_k)$ converging to p_k such that the first term of $T_{1,k}$ is k. Now let $H_i = \bigcup \{T_{i,k} | k \text{ an integer}\}$, $i = 0, 1$.

The set $E' = E - (\{r_k | k \text{ an integer}\} \cup \{p_k | k \text{ an integer}\})$ is countable, so let $e_1, e_2, \cdots$ be an enumeration of the points of E'. Let I_1 be a bounded open interval about e_1 such that $I_1 \cap (H_0 \cup H_1) = \emptyset$. Let T_1 be a sequence in $I_1 \cap D$ converging to e_1. Suppose T_p has been chosen for $p < n$. Let I_n be a bounded open interval about e_n such that

$$I_n \cap [(H_0 \cup H_1) \cup (\bigcup \{T_p | p < n\})] = \emptyset.$$

Let T_n be a sequence in $I_n \cap D$ converging to e_n. Let $Y = E \cup H_0 \cup H_1 \cup (\bigcup \{T_n | n = 1, 2, \cdots\})$, and let $D' = D \cap Y$.

Let $\tau_1 = T|Y$ and $\tau_2 = T'|Y$. Let τ_3 be the topology on Y generated by the following sets.

(i) $\{x\}$, $x \in E$,

(ii) cofinite subsets of Y,

(iii) $K_0 = H_0 \cup \{(Y - H_1) \cap (\bigcup \{[p_k, r_k) | k \text{ an integer}\})\}$,

$K_1 = H_1 \cup \{(Y - H_0) \cap (\bigcup \{[r_k, p_{k+1}) | k \text{ an integer}\})\}$,

$M_i = T_i \cup \{(Y - I_i) \cap E\}$, $i = 1, 2, \cdots$.

The proof given in [4] will apply to show that τ_1 and τ_2 are T_1-complements, and an argument similar to that will show that τ_1 and τ_3 are T_1-complements.

It is obvious that $\sup\{\tau_2, \tau_3\}$ is discrete. Let us show that if $U \in \tau_2 \cap \tau_3$ and $U \neq \emptyset$, then U is cofinite. Note that if $U \in \tau_3$ and $U \cap H_0 \neq \emptyset$ ($U \cap H_1 \neq \emptyset$) then U contains a cofinite subset of K_0 (K_1). Also $K_0 \cup K_1 = Y$. Hence if $U \cap H_0 \neq \emptyset$ and $U \cap H_1 \neq \emptyset$, then U is cofinite. Similarly if $U \in \tau_2$, $U \cap A_0 \neq \emptyset$, and $U \cap A_1 \neq \emptyset$, then U is cofinite. So to show $U \in \tau_2 \cap \tau_3$, $U \neq \emptyset$, implies U is cofinite, we need only show U intersects both H_0 and H_1 or that U intersects both A_0 and A_1.

First of all, let $U \in \tau_2 \cap \tau_3$, $U \neq \emptyset$. Then $U \cap E \neq \emptyset$ and $U \cap D' \neq \emptyset$. Let $x \in U \cap D'$ and $y \in U \cap E$. Then either $x \in H_0$, $x \in H_1$, or $x \in T_i$ for some i, and $y \in A_0$, $y \in A_1$, or $y \in S_i$ for some i.

Case 1. Let $x \in H_0$ and $y \in A_0$. Then U contains a cofinite subset of K_0. Hence U contains all but finitely many of the p_k's. But $p_k \in S_{1,k} \subset A_1$. So U intersects both A_0 and A_1, and therefore U is cofinite on Y. A similar argument works if $x \in H_1$ and $y \in A_1$.

Case 2. Let $x \in H_1$ and $y \in A_0$. Then U contains a cofinite subset of $B_0 \cap Y$ and a cofinite subset of K_1. Hence for some integer k, U contains all points of D' in $[k+\frac{1}{2}, k+1)$ and all points of E in $[r_k, p_{k+1})$. Hence U contains all points of Y in $[k+\frac{1}{2}, p_{k+1})$, and therefore $U \cap T_{0,k} \neq \emptyset$. Hence $U \cap H_0 \neq \emptyset$ and U is cofinite. A similar argument works if $x \in H_0$ and $y \in A_1$.

Case 3. Let $x \in H_0$ and $y \in S_i$ for some i. The U contains either a cofinite subset of $C_i \cap B_0 \cap Y$ or a cofinite subset of $C_i \cap B_1 \cap Y$. A cofinite subset of $C_i \cap B_1 \cap Y$ contains all but finitely many integers and each integer is in H_1. A cofinite subset of $C_i \cap B_0 \cap Y$ contains all points of D' in $[k-\frac{1}{2}, k)$ for some integer k. But $p_k \in [k-\frac{1}{2}, k)$, and $T_{1,k} \cap [k-\frac{1}{2}, k) \neq \emptyset$. Hence in both cases $C_i \cap B_j \cap Y$ contains points of H_1. So U intersects both H_0 and H_1 and is therefore cofinite. The cases for $x \in H_1$ and $y \in S_i$, $x \in T_i$ and $y \in A_0$, and $x \in T_i$ and $y \in A_1$ can be taken care of by similar arguments.

Case 4. Let $x \in T_i$ and $y \in S_j$ for some i and some j. Then U contains either a cofinite subset of $M_i \cap K_0$ or a cofinite subset of $M_i \cap K_1$. This implies that either U contains $p_k \in S_{1,k} \subset A_1$, or U contains $r_k \in S_{0,k} \subset A_0$ for some k. Also U contains either a cofinite subset of $C_j \cap B_0 \cap Y$ or a cofinite subset of $C_j \cap B_1 \cap Y$. Then, as above, either $U \cap H_0 \neq \emptyset$ or $U \cap H_1 \neq \emptyset$. Then applying Case 1 or Case 2 as is appropriate it follows that U is cofinite.

Therefore $U \in \tau_2 \cap \tau_3$, $U \neq \emptyset$, implies U is cofinite, and hence τ_2 and τ_3 and T_1-complements.

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