ON THE NEAR EQUICONTINUITY OF TRANSFORMATION GROUPS

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Let X be a compact metric space and h an auto-homeomorphism of X. It was proved in [2] that if h has equicontinuous powers at every $x \in X$ except at a finite set A, and if $X-A$ is connected, then the number of points of A is at most two, and each point of A is fixed under h. Now let X be a locally connected, locally compact metric space. Kaul [4] proved that if h has equicontinuous powers at every $x \in X$, except at a compact 0-dimensional subset A, and if $X-A$ is connected, then the number of points of A is at most two, and each point of A is fixed under h. Here we study the above theorem of Kaul in the setting of transformation groups. After proving a rather general theorem, we will generalize Kaul's theorem (Corollary 3).

If X is a metric space, d is any metric compatible with the topology of X, $x \in X$, and $\epsilon > 0$, we define

$$B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

If $A \subset X$, $Fr(A)$ denotes the boundary of A. Let (X, T, π) be a transformation group. We say that T is equicontinuous at $x \in X$ if for each $\epsilon > 0$ there is a $\delta > 0$ such that $[d(x, y) < \delta$ implies $d(xt, yt) < \epsilon]$ for every $t \in T$. A subset A of X is a minimal set if

$$A \neq \emptyset \quad \text{and} \quad A = \overline{xT} \qquad \text{for every } x \in A.$$

Our reference for dimension theory is the book [3].

THEOREM. *Let (X, T, π) be a transformation group, where X is a locally connected, locally compact metric space. Let $E(T)$ denote the set of points at which T is equicontinuous and $N(T) = X - E(T)$. Let $E(T)$ be connected and $N(T)$ be compact and zero dimensional. Then $N(T)$ contains at most two minimal sets.*

(*Note*: Since $N(T)$ is compact, nonempty, and T-invariant, by [1, 2.22, p. 15], $N(T)$ contains at least one minimal set.)

PROOF. Let $s = \{s_n : n \in Z^+\}$ be a fixed sequence in T. If $x \in X$, let $C(x, s)$ denote the set of cluster points of the sequence $\{xs_n\}$. We define

$$A = \{x \in E(T) : C(x, s) \cap E(T) \neq \emptyset\}.$$

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We will prove

(1) A is open and closed in $E(T)$.

Choose $z\in A$ and let $x\in C(z, s)\cap E(T)$. Choose $\epsilon>0$ such that

$$\overline{B(x, \epsilon)} \cap N(T) = \emptyset \quad \text{and} \quad \overline{B(x, \epsilon)}$$

is compact. Choose $\delta>0$ such that $d(z, y)<\delta$ implies $d(zt, yt)<\epsilon/2$ for every $t\in T$, and $B(z, \delta)\subset E(T)$. Since $x\in C(z, s)$, we may select a subsequence u of s such that $d(x, zu_n)<\epsilon/2$ for every n. If $y\in B(z, \delta)$, then

$$d(x, yu_n) \leqq d(x, zu_n) + d(zu_n, yu_n) < \epsilon$$

for every n. Thus

$$yu_n \in \overline{B(x, \epsilon)}$$

for every n, hence the sequence $\{yu_n\}$ has a cluster point

$$w \in \overline{B(x, \epsilon)}.$$

Then $w\in C(y, s)\cap E(T)$ and $B(z, \delta)\subset A$. We have shown that A is open in $E(T)$.

We now prove that A is closed in $E(T)$. Let $x\in \overline{A}\cap E(T)$, and choose $\epsilon>0$ such that

$$\overline{B(x, \epsilon)} \cap N(T) = \emptyset \quad \text{and} \quad \overline{B(x, \epsilon)}$$

is compact. Choose $\delta>0$ such that $d(x, w)<\delta$ implies $[d(xt, wt)<\epsilon/3]$ for every $t\in T$. Since $x\in\overline{A}$, we may choose $y\in B(x, \delta)\cap A$, $z\in C(y,s)\cap E(T)$, and $\delta_1>0$ such that $d(z, w)<\delta_1$ implies $d(zt, wt)<\epsilon/6$ for every $t\in T$. Choose a subsequence u of s such that $d(z, yu_n)<\delta_1$ for every n. Then

$$d(y, yu_nu_1^{-1}) \leqq d(y, zu_1^{-1}) + d(zu_1^{-1}, yu_nu_1^{-1}) < \epsilon/3$$

for every n. Thus

$$d(x, xu_nu_1^{-1}) \leqq d(x, y) + d(y, yu_nu_1^{-1}) + d(yu_nu_1^{-1}, xu_nu_1^{-1}) < \epsilon$$

for every n, hence

$$xu_nu_1^{-1} \in \overline{B(x, \epsilon)} \qquad \text{for every } n.$$

Hence there is a subsequence $\{u_{n_k}\}$ of u such that

$$\{xu_{n_k}u_1^{-1}\} \to w \in \overline{B(x, \epsilon)}.$$

Then $\{xu_{n_k}\}\to wu_1$, and since $E(T)$ is invariant under T, we have

$$wu_1 \in C(x, s) \cap E(T)$$

so that $x \in A$. We have now shown that A is closed in $E(T)$.

Now let $z \in N(T)$. Since T is not equicontinuous at z, there is an $\epsilon > 0$ such that for every $\delta > 0$, diam $B(z, \delta)t > \epsilon$ for some $t \in T$. Choose a finite open cover $U_1, \cdots, U_n$ of $N(T)$ such that the closures of the U_i are compact and pairwise disjoint, diam $U_i < \epsilon$, and $N(T) \cap \mathrm{Fr}(U_i) = \varnothing$ for every i. Let $\{V_k; k \in Z^+\}$ be a sequence of connected open neighborhoods of z such that diam $V_k < 1/k$ and let $U' = U_1 \cup \cdots \cup U_n$. For each k there is a $t_k \in T$ such that diam $V_k t_k^{-1} > \epsilon$; we note that $V_k t_k^{-1}$ meets some U_i but can be contained in no U_i. Since $V_k t_k^{-1}$ is connected, there is an $x_k \in V_k$ such that for some i, $x_k t_k^{-1} \in \mathrm{Fr}(U')$. Since $\mathrm{Fr}(U')$ is compact, we may assume that $\{x_k t_k^{-1}\} \to x \in \mathrm{Fr}(U')$. We prove that $\{x t_k\} \to z$. Let $\delta > 0$ be given. Since $x \in E(T)$, there is a $\delta_1 > 0$ such that $d(x, y) < \delta_1$ implies $d(xt, yt) < \delta/2$ for every $t \in T$; choose m so large that $[d(x_k t_k^{-1}, x) < \delta_1]$ and $1/k < \delta/2$ for every $k \geqq m$. Then if $k \geqq m$ we have

$$d(z, xt_k) \leqq d(z, x_k) + d(x_k, xt_k) < \delta.$$

Now if we let $t = \{t_k\}$ we get from (1) that

(2) $C(x, t) \cap E(T) = \varnothing$ for every $x \in E(T)$.

Now let U be an open neighborhood of z such that $\overline{U}$ is compact and $N(T) \cap \mathrm{Fr}(U) = \varnothing$. Let

$$B = \{x \in E(T) \colon \{xt_n\} \text{ is eventually in } U\}.$$

We will prove that B is open and closed in $E(T)$.

Let $x \in \overline{B} \cap E(T)$ and assume that $x \notin B$. Then there is a subsequence u of t such that $xu_n \notin U$ for every n. Since T is equicontinuous at x, there is an $x_k \in B$ such that $d(x_k u_n, xu_n) < 1/k$ for every n. Since $\{x_k u_n\}$ is eventually in U we may choose a subsequence $\{u_{n_k}\}$ of u such that

(3) $d(x_k u_{n_k}, xu_{n_k}) < 1/k$ for every k and $x_k u_{nk} \in U$. Then $\{x_k u_{n_k}\}$ clusters at a point $y \in \overline{U}$. By (3), $y \in C(x, u)$, hence $y \in \mathrm{Fr}(U)$. Therefore we have $y \in \mathrm{Fr}(U) \cap C(x, t)$, and this contradicts (2). We have shown that B is closed in $E(T)$.

Let $x \in B$ and assume $x \notin \mathrm{int}(B)$. Then for each k choose an open neighborhood $U_k \subset E(T)$ of x such that $y \in U_k$ implies $d(xs, ys) < 1/k$ for every $s \in T$. Since $U_k - B \neq \varnothing$, there is a $y_k \in U_k$ such that $y_k t_n \notin U$ for infinitely many n. Since $x \in B$, we may select a subsequence u of t such that $y_k u_k \notin U$ and $xu_k \in U$ for every k. Then $\{xu_k\}$ clusters at a point $[w \in \overline{U}]$. Since $d(xu_k, y_k u_k) < 1/k$ for every k, $\{y_k u_k\}$ clusters at w also. Thus $w \in \mathrm{Fr}(U)$ and hence $C(x, t) \cap E(T) \neq \varnothing$; this contradicts (2), and shows that B is open. Finally, since $B \neq \varnothing$, we find that $B = E(T)$. Since each point of $N(T)$ has a neighborhood base consisting of open sets with compact closures whose boundaries do not meet $N(T)$, we have shown that

(4) If $z \in N(T)$, there exists a sequence t in T such that $\{xt_n\} \to z$ for each $x \in E(T)$.

We now show that $N(T)$ contains at most two minimal sets. Assume that x_1, x_2, and $x_3 \in N(T)$ are such that

$$\overline{x_1T},\ \overline{x_2T},\quad \text{and} \quad \overline{x_3T}$$

are distinct and minimal. Since minimal sets are disjoint we must have

$$\overline{x_1T} \cap \overline{x_2T} = \overline{x_1T} \cap \overline{x_3T} = \overline{x_2T} \cap \overline{x_3T} = \varnothing.$$

Since $N(T)$ is compact and zero dimensional and X is locally compact, we may choose an open subset W of X such that

$$\overline{x_1T} \subset W, \qquad \overline{W} \text{ is compact, } \overline{W} \cap (\overline{x_2T} \cup \overline{x_3T}) = \varnothing,$$

and $N(T) \cap \operatorname{Fr}(W) = \varnothing$. Choose connected open neighborhoods V and U of x_1 and x_2, respectively, such that $V \subset W$, $\overline{U}$ is compact, and

$$\overline{U} \cap \overline{x_3T} = \varnothing.$$

Since $E(T)$ is dense in X, we may select $y \in U \cap E(T)$. By (4), there is an $s_1 \in T$ such that $ys_1 \in V$. Then $C = \overline{V} \cup \overline{U}s_1$ is a compact connected set such that

$$C \cap \overline{x_1T} \neq \varnothing, \qquad C \cap \overline{x_2T} \neq \varnothing,$$

and

$$C \cap \overline{x_3T} = \varnothing.$$

Now choose an open neighborhood M of x_3 such that the closure of M does not meet the set C. Choose an $\epsilon > 0$ such that $\overline{M} \cap B(C, \epsilon) = \varnothing$, and let s be any sequence in T. Then for each integer n we have that

$$\overline{x_1T} \cap Cs_n^{-1} \neq \varnothing$$

and

$$\overline{x_2T} \cap Cs_n^{-1} \neq \varnothing,$$

and since Cs_n^{-1} is connected, we must have $Cs_n^{-1} \cap \operatorname{Fr}(W) \neq \varnothing$. Thus for each n, we have $y_n \in C$ such that $y_ns_n^{-1} \in \operatorname{Fr}(W)$. Since $\operatorname{Fr}(W)$ and C are compact, there is a subsequence $\{s_{n_k}\}$ of s such that $\{y_{n_k}s_{n_k}^{-1}\} \to y \in \operatorname{Fr}(W)$ and $\{y_{n_k}\} \to x \in C$. Because $y \in E(T)$, we may select $\delta > 0$ such that if $d(w, y) < \delta$ then $d(wu, yu) < \epsilon/2$, all $u \in T$. Choose m so large that $k \geqq m$ implies $d(y, y_{n_k}s_{n_k}^{-1}) < \delta$ and $d(y_{n_k}, x) < \epsilon/2$. Then if

$k \geqq m$, we have

$$d(x, ys_{n_k}) \leqq d(x, y_{n_k}) + d(y_{n_k}, ys_{n_k}) < \epsilon.$$

This means $ys_{n_k} \in B(C, \epsilon)$ for every $k \geqq m$. Thus the sequence $\{ys_n\}$ cannot eventually be in M. Since s is an arbitrary sequence in T, this contradicts (4) and completes the proof.

In the following corollaries the hypothesis of the theorem is assumed.

COROLLARY 1. *If x_1, x_2, and $x_3 \in N(T)$, then of the three orbit closures*

$$\overline{x_1 T}, \quad \overline{x_2 T}, \quad \text{and} \quad \overline{x_3 T},$$

at least two have a nonempty intersection.

PROOF. Each $\overline{x_i T}$ contains a minimal set [**1**, 2.22, p. 15]. The result follows from the theorem.

COROLLARY 2. *If T is almost periodic at each point of $N(T)$, then $N(T)$ is either minimal or is the union of two minimal sets.*

PROOF. By [**1**, 4.07, p. 31], each orbit closure of a point in $N(T)$ is minimal. The result follows from the theorem.

COROLLARY 3. *If T is abelian or connected, then each point of $N(T)$ is fixed under T and $N(T)$ contains at most two points.*

PROOF. If T is connected, the orbit of each point of $N(T)$ is a connected subset of $N(T)$, hence is a point. If T is abelian, let $z \in N(T)$ and assume there is an $s \in T$ such that $z \neq zs$. Then there is an open neighborhood U of z such that U is compact, $N(T) \cap \mathrm{Fr}(U) = \emptyset$, and $U \cap Us = \emptyset$. By (4), there is a sequence t in T such that $\{xt_n\}$ is eventually in U for every $x \in E(T)$. Then $\{xt_n\}$ and $\{xs^{-1}t_n\}$ are eventually in U, hence there is an integer m such that $xt_m, xs^{-1}t_m \in U$. But this means that $xt_m \in U \cap Us$, and contradicts the choice of U. Therefore z is fixed under T.

We have shown that T leaves every point of $N(T)$ fixed. Thus each point of $N(T)$ is a minimal set, and the corollary follows from the theorem.

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