ISOMETRIES OF THE TRACE CLASS
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Let $\mathcal{S}$ denote the Banach space of trace class operators on a complex Hilbert space $H$, in the norm $\|T\|_1 = \text{Tr}(|T|)$. The space $\mathcal{S}$ is a two-sided ideal in the algebra $\mathcal{L}$ of all bounded operators on $H$. See [4].

**Theorem.** If $\Phi$ is a linear isometry of the Banach space $\mathcal{S}$ onto itself, then there exists a $*$-automorphism or a $*$-antiautomorphism $\alpha$ of $\mathcal{L}$ and a unitary operator $U$ in $\mathcal{L}$ such that $\Phi(T) = \alpha(TU)$, ($T$ in $\mathcal{S}$).

**Remark 1.** The theorem provides a partial answer to [3, Remark 1, p. 231].

**Proof.** The adjoint $\Phi'$ is a linear isometry of $\mathcal{L}$ onto $\mathcal{L}$ so by results of Kadison [2, Theorem 7, Corollary 11] has the form $\Phi'(A) = U\alpha(A)$ where $\alpha$ and $U$ are as described in the statement of the theorem. It is elementary that $\Phi(T) = \Psi(TU)$ where $\Psi = \alpha$. The proof will be complete if it is shown that $\alpha$ is the adjoint of $\alpha^{-1}$ (restricted to $\mathcal{S}$). By the folk result [1, pp. 256, 9] it is sufficient to check this in the following two cases:

(i) $\alpha(A) = VA^{-1} V$ with $V$ a fixed unitary operator; then $\langle T, \alpha(A) \rangle = \langle T, VA^{-1} V \rangle = \langle A^{-1} TV, A \rangle = \langle \alpha^{-1}(T), A \rangle$,

(ii) after the choice of an orthonormal basis, $\alpha(A)$ is the transposed matrix of $A$; then $\langle T, \alpha(A) \rangle = \text{Tr}(T\alpha(A)) = \text{Tr}(\alpha(T)A) = \langle \alpha^{-1}(T), A \rangle$.

**Remark 2.** A previous version of the above proof exploited a knowledge of the extreme points of the unit sphere of $\mathcal{S}$. These were determined to be the partial isometries with initial (hence final) domain one-dimensional.

**References**


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