A GRONWALL INEQUALITY FOR LINEAR
STIELTJES INTEGRALS

J. V. HEROD

This paper provides a Gronwall type inequality which includes the one found by Schmaedeke and Sell \[4\].

Suppose that \( S \) is an interval of real numbers containing zero and \( OB \) is the collection of functions from \( S \) to the real numbers each member of which is of bounded variation on each finite interval of \( S \). The numeral 1 will also denote the constant function from \( S \) which has only the value 1; if \( x \) is in \( S \), then \( 1_x \) denotes the function from \( S \) which has the value 1 at \( x \) and the value 0 elsewhere; and \( 0_x \) denotes the function \( 1 - 1_x \). Let \( J \) be a function from \( OB \) to the collection of functions from \( S \times S \) to the real numbers having the following properties: if each of \( f \) and \( g \) is in \( OB \) and \( \{x, y, z\} \) is in \( S \times S \times S \) then

\[
\begin{align*}
(1) \quad J[f](x, y) + J[g](x, y) &= J[f + g](x, y), \\
(2) \quad J[r \cdot f](x, y) &= r \cdot J[f](x, y), \\
(3) \quad J[f](x, y) + l[f](y, z) &= J[f](x, z) \quad \text{provided that } x \leq y \leq z \text{ or } x \geq y \geq z, \\
(4) \quad J[f](x, z) \geq 0 \quad \text{provided that } f(y) \geq 0 \text{ for } x \leq y \leq z \text{ or } x \geq y \geq z, \\
\end{align*}
\]

and

\[
\begin{align*}
(5) \quad \text{if } x \text{ is in } S \text{ and } x \geq 0 \text{ then each of } J[0_x](x, x^+) \text{ and } J[1_x](x^-, x) \text{ is less than 1; whereas, if } x \text{ is in } S \text{ and } x \leq 0 \text{ then each of } J[1_x](x^+, x) \text{ and } J[0_x](x, x^-) \text{ is less than 1.}
\end{align*}
\]

Theorem. If \( J \) satisfies properties (1)–(5), there is a function \( m \) from \( S \times S \) to the real numbers having the following properties:

\[
\begin{align*}
(i) \quad m(x, y) &\geq 1 \text{ for each } \{x, y\} \text{ in } S \times S, \\
(ii) \quad m(x, y) \cdot m(y, z) &= m(x, z) \quad \text{provided that } x \leq y \leq z \text{ or } x \geq y \geq z, \\
(iii) \quad m(0, x) = 1 + J[m(0, \cdot)](0, x) \text{ for each } x \text{ in } S, \text{ and} \\
(iv) \quad \text{if } f \text{ is in } OB, P \text{ is a number, and } f(x) \leq P + J[f](0, x) \text{ for each } x \text{ in } S, \text{ then } f(x) \leq P \cdot m(0, x) \text{ for each } x \text{ in } S.
\end{align*}
\]

Remark. It is the purpose of this remark to show a connection between the above theorem and one of Schmaedeke and Sell. In \[4\], they investigate an inequality similar to that in part (iv) but use the mean Stieltjes integral and the Dushnik or interior integral (see also \[3\]). One-term approximating sums for these are indicated:

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\[
(M) \int_x^y f dg \sim \frac{f(x) + f(y)}{2} \cdot [g(y) - g(x)]
\]

and

\[
(I) \int_x^y f dg \sim f(z) \cdot [g(y) - g(x)] \text{ where } x < z < y \text{ or } x > z > y.
\]

If no member of \( S \) is negative, \( g \) is increasing and right continuous, and, for \( x \leq y \), \( J[f](x, y) \) is defined to be \((M) \int_x^y f dg\), with \( J[f](y, x) = J[f](x, y) \), then \( J \) satisfies properties (1)-(4) and, also, property (5) in case \( g(z) - g(z^-) < 2 \) for all \( z \) different from zero. If \( P \) is a number, \( f \) is in \( OB \), and \( f(x) \leq P + (M) \int_x^y f dg \) for each \( x \) in \( S \), then \( f(x) \leq P + J[f](0, x) \) for each \( x \) in \( S \), since no member of \( S \) is negative. This inequality includes the inequality of [4, p. 1219]. If, instead, \( J \) is defined in terms of the interior integral then properties (1)-(4) are, again, satisfied by \( J \) and property (5) makes no additional requirement due to the condition that \( g \) is right continuous. (See Remark 1 of [4].)

Remark. With properties (1)-(3), a more familiar property which is equivalent to the conceptually simpler property (4) is

\[(4') \text{ if } f \text{ is in } OB \text{ and } \{x, z\} \text{ is in } S \times S \text{ and } m \text{ is a number such that } |f(y)| \leq m \text{ for all } y \text{ in } S \text{ such that } x \leq y \leq z \text{ or } x \geq y \geq z, \text{ then } |J[f](x, z)| \leq mJ[1](x, z) \text{ (compare [2, Axiom II]).}
\]

To see that (4) implies \((4')\), notice that each of \( m + f(y) \) and \( m - f(y) \) is nonnegative for \( x \leq y \leq z \) or \( x \geq y \geq z \); to see that \((4')\) implies (4), notice that each of \( J[1_x] \) and \( J[0_x] \) has only nonnegative values and use the formulas in Theorem 1 and equation (24) of [2]. We shall use the fact that if \( f \) is in \( OB \) and \( \{x, y\} \) is in \( S \times S \) then \( J[f](x, y) \leq J[|f|](x, y) \) (which follows from properties (1)-(4).

Indication of Proof of Theorem. The proof of parts (i), (ii), and (iii) of the theorem is only a slight modification of the ideas developed by MacNerney in [2, Theorems 1 and 2] and used by the author in [1, Theorem 1.1]. For part (iv), suppose that \( f \) is in \( OB \), \( P \) is a number, and \( f(x) \leq P + J[f](0, x) \) for each \( x \) in \( S \). Define a sequence \( h \) with values in \( OB \) as follows: \( h_0 = f \) and, if \( n \) is a positive integer, then \( h_n(x) = P + J[h_{n-1}](0, x) \) for each \( x \) in \( S \). Let \( r \) be a function so that if \( x \) is in \( S \) then \( r(x) = \int_0^x d[h_2 - h_1] \). Let \( L \) be a sequence so that if \( x \) is in \( S \) then \( L_1(x) = r(x) \) and if \( n \) is a positive integer then \( L_{n+1}(x) = J[L_n](0, x) \). If \( n \) is a positive integer and \( a \) is in \( S \), then

\[
0 \leq \sum_{p=1}^n L_p(a) \leq \sum_{p=1}^{n+1} L_p(a) \leq r(a) \cdot m(0, a).
\]
Moreover, if \( x \) is in \( S \) and between 0 and \( a \) and \( n \) is a positive integer then \( L_n(x) \leq L_n(a) \). Finally, if \( n \) is a positive integer and \( x \) is in \( S \) then 
\[
|h_{n+1}(x) - h_n(x)| \leq L_n(x).
\]
Thus the sequence \( h \) converges absolutely and, if \( a \) is in \( S \), uniformly on the set of all numbers in \( S \) between 0 and \( a \). Moreover, if \( \lim h = U \) and \( a \) is in \( S \), then \( U(a) = P + \int [U](0, a) \); and \( U(x) = pm(0, x) \) for each \( x \) in \( S \). (To see this latter, recall [2, Theorems 2 and E].) We have, inductively, that if \( p \) is a positive integer and \( x \) is in \( S \) then \( f(x) \leq h_p(x) \). Consequently, \( f(x) \leq pm(0, x) \).

**Remark.** Using [1, Lemma 1.1] and similar techniques to the ones indicated above, we may obtain a more general inequality for a function \( f \) which satisfies \( f(x) \leq P + \int [f](0, x) + g(x) \) where \( g \) is in \( OB \) and \( g(0) = 0 \).

**Bibliography**


**Georgia Institute of Technology**