REMARKS ON THE STRING OF BEADS

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1. Introduction. Let $D$ be the closed unit disc in the complex plane $C$, $D = \{ z : |z| \leq 1 \}$. Let $K$ be a closed subset of the interval $[-1, 1]$. If $I$ is any interval component of $[-1, 1] - K$, remove from $D$ the open disc with $I$ as a diameter. Let $X$ be the compact set which remains when all such open discs have been deleted. In the case where $K$ has no interior relative to $[-1, 1]$, $X$ is referred to as the "string of beads" example. Let $R(X)$ denote the function algebra consisting of the functions which are uniform limits on $X$ of rational functions with poles off $X$. The string of beads example has been studied with regard to the question of the connectivity of the Gleason parts of $R(X)$ for an arbitrary compact plane set $X$. We refer the reader to [1], [4] and [5] for more detailed discussions of all of the concepts discussed below.

It can be shown (an outline of the proof is in §2) that, when $K$ has zero linear measure, the upper and lower components of the interior of $X$, denoted $U$ and $L$, respectively, lie in distinct parts of $R(X)$. Moreover, each point of $K$ is a peak point for $R(X)$. When $K$ has positive linear measure it is easy to see that both $U$ and $L$ lie in the same part. In [3] Hoffman conjectured that in this case there is always some point of $K$ which is in the same part as $U$ and $L$ so that the part is connected. Since it is known that when a point of $K$ fails to be a peak point then it belongs to the nontrivial part of $R(X)$, it suffices to show some point of $K$ is not a peak point. The purpose of this note is to show that in many cases Hoffman's conjecture is correct and that, moreover, in these cases, almost every point of $K$ belongs to the nontrivial part.

The remarkable new fact, however, is that A. M. Davie of the University of Dundee, Dundee, Scotland has constructed a string of beads example for which $K$ has positive linear measure but each point of $K$ is a peak point. Hence $U$ and $L$ separate the part containing them and exhibit the first example of a disconnected part for $R(X)$. John Garnett of UCLA has constructed a similar example independently. With these startling examples in mind, we record our theorem with its proof merely as an interesting observation about the string of beads examples.

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2. Some details. As promised in the introduction we begin with a sketch of the proof of the fact that, when $K$ has zero linear measure, $U$ and $L$ lie in different parts. It follows from a theorem of Melnikov (see [5, p. 112, Theorem 14.1]) that for any string of beads example, $R(X)=A(X)$, where $A(X)$ denotes the algebra of all continuous functions of $X$ which are analytic on the interior of $X$. But it is an easy consequence of a theorem of Fatou and its proof (see [2, p. 80]) that there is a function in $A(X)$ with $f=1$ on $U$ and $|f|<1$ on $L$. The existence of such a function in $R(X)$ is sufficient to distinguish the parts containing $U$ and $L$. When $K$ has positive linear measure, harmonic measure for a point in $U$ is not singular with respect to harmonic measure for a point in $L$—hence $L$ and $U$ lie in the same part (see Chapter VI of [1]).

Our observation about the string of beads is the following. For the $n$th deleted disc, let $r_n$ denote its radius, let $C_n$ denote its boundary, and let $x_n$ denote its center.

**Theorem.** If $\sum_{n=1}^{\infty} r_n < 1$ and $\sum_{n=1}^{\infty} \sqrt{r_n} < \infty$, then almost every point of $K$ with respect to linear measure is not a peak point for $R(X)$. (The first condition is, of course, equivalent to $K$ having positive linear measure.)

**Corollary.** Under the hypotheses of the theorem, the part containing $U$ and $L$ is connected and contains almost every point of $K$.

**Proof of Corollary.** A point of $X$ is not a peak point if and only if the part containing it has positive planar Lebesgue measure (see [4]).

To prove the theorem we use the following well-known lemma.

**Lemma.** Let $t \in K$. Let $a_n = |t-x_n|$. If $\sum_{n=1}^{\infty} r_n/a_n < \infty$, then $t$ is not a peak point of $R(X)$.

**Proof of Lemma.** To show $t$ is not a peak point it suffices to exhibit a nontrivial finite (complex) representing measure for $t$. Define a measure $\mu$ on the boundary of $X$ by

$$\mu = \frac{1}{2\pi i} \frac{dz}{z-t} \quad \text{on } |z| = 1$$

and

$$\mu = \frac{1}{2\pi i} \frac{-dz}{z-t} \quad \text{on } C_n, n = 1, 2, \ldots.$$
It is an elementary computation to show the condition

$$\sum_{n=1}^{\infty} \frac{r_n}{a_n} < \infty$$

guarantees that $\mu$ is a finite measure. The fact that it represents is an immediate consequence of the Cauchy integral formula.

**Proof of Theorem.** Let $\epsilon > 0$ satisfy $\sum_{n=1}^{\infty} r_n + \epsilon < 1$. Since $\sum_{n=1}^{\infty} \sqrt{r_n} < \infty$ we can choose positive numbers $b_n$ such that $b_n \leq 1$ for all $n$,

$$\sum_{n=1}^{\infty} \frac{r_n}{b_n} < \sum_{n=1}^{\infty} r_n + \epsilon \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty.$$  

Let $K_\epsilon$ be the set left in $[-1, 1]$ after removing from $D$ discs of radius $r_n/b_n$ with centers $x_n$. Then $K_\epsilon \subset K$ and almost every point of $K$ lies in some $K_\epsilon$, where $0 < \epsilon < 1 - \sum_{n=1}^{\infty} r_n$. Now if $t \in K_\epsilon$, some $\epsilon$, then $|t - x_n| \geq r_n/b_n$. Hence, if $a_n = |t - x_n|$, $\sum_{n=1}^{\infty} (r_n/a_n) \leq \sum_{n=1}^{\infty} b_n < \infty$. By the lemma $t$ is not a peak point of $R(X)$.

**Remark.** Since the end points of the intervals complementary to $K$ are peak points, “almost every” cannot be replaced by “every” in the statement of the theorem.

**References**


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