REMARKS ON THE STRING OF BEADS

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1. **Introduction.** Let D be the closed unit disc in the complex plane C, $D=\{z: |z| \leqq 1\}$. Let K be a closed subset of the interval $[-1, 1]$. If I is any interval component of $[-1, 1]-K$, remove from D the open disc with I as a diameter. Let X be the compact set which remains when all such open discs have been deleted. In the case where K has no interior relative to $[-1, 1]$, X is referred to as the "string of beads" example. Let $R(X)$ denote the function algebra consisting of the functions which are uniform limits on X of rational functions with poles off X. The string of beads example has been studied with regard to the question of the connectivity of the Gleason parts of $R(X)$ for an arbitrary compact plane set X. We refer the reader to [1], [4] and [5] for more detailed discussions of all of the concepts discussed below.

It can be shown (an outline of the proof is in §2) that, when K has zero linear measure, the upper and lower components of the interior of X, denoted U and L, respectively, lie in distinct parts of $R(X)$. Moreover, each point of K is a peak point for $R(X)$. When K has positive linear measure it is easy to see that both U and L lie in the same part. In [3] Hoffman conjectured that in this case there is always some point of K which is in the same part as U and L so that the part is connected. Since it is known that when a point of K fails to be a peak point then it belongs to the nontrivial part of $R(X)$, it suffices to show some point of K is not a peak point. The purpose of this note is to show that in many cases Hoffman's conjecture is correct and that, moreover, in these cases, almost every point of K belongs to the nontrivial part.

The remarkable new fact, however, is that A. M. Davie of the University of Dundee, Dundee, Scotland has constructed a string of beads example for which K has positive linear measure but each point of K is a peak point. Hence U and L separate the part containing them and exhibit the first example of a disconnected part for $R(X)$. John Garnett of UCLA has constructed a similar example independently. With these startling examples in mind, we record our theorem with its proof merely as an interesting observation about the string of beads examples.

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2. **Some details.** As promised in the introduction we begin with a sketch of the proof of the fact that, when K has zero linear measure, U and L lie in different parts. It follows from a theorem of Melnikov (see [**5**, p. 112, Theorem 14.1]) that for any string of beads example, $R(X)=A(X)$, where $A(X)$ denotes the algebra of all continuous functions of X which are analytic on the interior of X. But it is an easy consequence of a theorem of Fatou and its proof (see [**2**, p. 80]) that there is a function in $A(X)$ with $f\equiv 1$ on U and $|f|<1$ on L. The existence of such a function in $R(X)$ is sufficient to distinguish the parts containing U and L. When K has positive linear measure, harmonic measure for a point in U is not singular with respect to harmonic measure for a point in L—hence L and U lie in the same part (see Chapter VI of [**1**]).

Our observation about the string of beads is the following. For the nth deleted disc, let r_n denote its radius, let C_n denote its boundary, and let x_n denote its center.

THEOREM. *If* $\sum_{n=1}^{\infty} r_n<1$ *and* $\sum_{n=1}^{\infty} \sqrt{r_n}<\infty$, *then almost every point of* K *with respect to linear measure is not a peak point for* $R(X)$. (*The first condition is, of course, equivalent to* K *having positive linear measure.*)

COROLLARY. *Under the hypotheses of the theorem, the part containing* U *and* L *is connected and contains almost every point of* K.

PROOF OF COROLLARY. A point of X is not a peak point if and only if the part containing it has positive planar Lebesgue measure (see [**4**]).

To prove the theorem we use the following well-known lemma.

LEMMA. *Let* $t\in K$. *Let* $a_n=|t-x_n|$. *If* $\sum_{n=1}^{\infty} r_n/a_n<\infty$, *then* t *is not a peak point of* $R(X)$.

PROOF OF LEMMA. To show t is not a peak point it suffices to exhibit a nontrivial finite (complex) representing measure for t. Define a measure μ on the boundary of X by

$$\mu = \frac{1}{2\pi i}\,\frac{dz}{z-t} \qquad \text{on } |z| = 1$$

and

$$\mu = \frac{1}{2\pi i}\,\frac{-dz}{z-t} \qquad \text{on } C_n,\ n = 1, 2, \cdots.$$

It is an elementary computation to show the condition

$$\sum_{n=1}^{\infty} \frac{r_n}{a_n} < \infty$$

guarantees that μ is a finite measure. The fact that it represents is an immediate consequence of the Cauchy integral formula.

PROOF OF THEOREM. Let $\epsilon > 0$ satisfy $\sum_{n=1}^{\infty} r_n + \epsilon < 1$. Since $\sum_{n=1}^{\infty} \sqrt{r_n} < \infty$ we can choose positive numbers b_n such that $b_n \leqq 1$ for all n,

$$\sum_{n=1}^{\infty} \frac{r_n}{b_n} < \sum_{n=1}^{\infty} r_n + \epsilon \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty.$$

Let K_ϵ be the set left in $[-1, 1]$ after removing from D discs of radius r_n/b_n with centers x_n. Then $K_\epsilon \subset K$ and almost every point of K lies in some K_ϵ, where $0 < \epsilon < 1 - \sum_{n=1}^{\infty} r_n$. Now if $t \in K_\epsilon$, some ϵ, then $|t - x_n| \geqq r_n/b_n$. Hence, if $a_n = |t - x_n|$, $\sum_{n=1}^{\infty} (r_n/a_n) \leqq \sum_{n=1}^{\infty} b_n < \infty$. By the lemma t is not a peak point of $R(X)$.

REMARK. Since the end points of the intervals complementary to K are peak points, "almost every" cannot be replaced by "every" in the statement of the theorem.

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