EMBEDDING A TRANSFORMATION GROUP IN AN AUTOMORPHISM GROUP

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1. Introduction. Using a construction of Baldosov [1], we show that a topological transformation group with completely regular phase space $X$ and locally compact phase group $T$ can be equivariantly embedded in a transformation group of automorphisms of a topological group $A$. The group $A$ in question is the free abelian topological group over $X$; some facts about $A$ are established in §2. In §4 several dynamical properties of $(A, T)$ are discussed as they relate to properties of $(X, T)$.

As general references to the notation and notions for transformation groups used here, see [4] and [5].

All topological spaces considered below, and in particular all topological groups, are assumed to be Hausdorff.

2. The topological group $A(X)$. Let $X$ be a completely regular space. Denote by $A(X)$ or simply $A$ the free abelian topological group over $X$ [6], [7, §8]. Algebraically $A$ is just the free abelian group generated by the set $X$; the topology of $A$ is the greatest separated topology compatible with the group structure and inducing on $X$ a topology weaker than the one initially given on $X$. We have:

(1) $A$ is a topological group containing $X$ as a closed subspace.

(2) If $f$ is a continuous map of $X$ into an abelian topological group $G$, then the unique extension of $f$ to a group homomorphism of $A$ into $G$ is continuous.

**Proposition 1.** Let $X$ be compact and infinite. Then $A$ is not a Baire space and a fortiori is not locally compact.

**Proof.** If $z \in A$ and $z \neq 0$, there exist distinct $x_1, \ldots, x_n \in X$ and nonzero integers $\alpha_1, \ldots, \alpha_n$ with $z = \sum_i \alpha_i x_i$, and we let $L(z) = \sum_i |\alpha_i|$. Set $L(0) = 0$. For each positive integer $n$ let $A_n = \{z \in A, L(z) \leq n\}$. Since $A = \bigcup_i A_i$, it is enough to show that each $A_n$ is compact and has vacuous interior.

Let $n > 0$. Let $B_0 = \{0\}$, and for each positive integer $p$ let $B_p$ be the set of all elements of $A$ of the form $\sum_i x_i$ where $x_1, \ldots, x_p \in X$, not necessarily distinct. Then $A_n = \bigcup \{B_p - B_q \mid 0 \leq p + q \leq n\}$, where

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Suppose $A_n$ contains a nonempty open subset $W$ of $A$. Choose $v \in W$ and let $U = -v + W$. Since $A$ is nondiscrete, there exists $z \in A$ with $z \neq 0$ and $(2n+1)z \in U$. Let $u = (2n+1)z$. Then $L(u) = (2n+1)L(z) \geq 2n+1$. On the other hand, $u = -v + w$ for some $w \in W$, whence $L(u) \leq L(v) + L(w) \leq 2n$.

**Proposition 2.** Let $\phi: X \to Y$ be a continuous surjection of compact spaces. Then the canonical map $\phi^*: A(X) \to A(Y)$ induced by $\phi$ is a continuous-open group epimorphism.

**Proof.** Let $N = \ker \phi^*$, $G = A(X)/N$. The map $\psi: G \to A(Y)$ associated with $\phi^*$ is continuous and is an algebraic isomorphism. To show that $\phi^*$ is open, we prove that $\psi$ is homeomorphic.

Let $B = Y\psi^{-1}$, and let $B_1$ be the image of $X$ under the projection of $A(X)$ onto $G$, so that $B_1 \subset B$. Now $B_1$ generates $G$ since $X$ generates $A(X)$, and $B$ is free in $G$ since $Y$ is free in $A(Y)$. Then $B_1 = B$. Since $B$ is closed in $G$, it follows that $G = A(B)$. Moreover, $\psi$ maps $B$ homeomorphically onto $Y$, so $\psi$ is the canonical map of $A(B)$ into $A(Y)$ induced by $\psi|B$. Hence $\psi$ is homeomorphic.

3. **The transformation group** $(A(X), T)$. Now let the completely regular space $X$ be the phase space of a transformation group $(X, T, \pi)$. Following Baidosov [1], we extend $\pi$ to an action $\pi^*$ of $T$ on $A = A(X)$ as follows. Let $j$ be the inclusion map of $X$ into $A$. For each $t \in T$ let $\pi^t$ be the continuous endomorphism of $A$ which extends the continuous map $\pi^tj$, where $\pi^t$ is the $t$-transition given by $x\pi^t = (x, t)\pi$ for $x \in X$. For $z \in A$ and $t \in T$ let $(z, t)\pi^* = z\pi^t$. We call $\pi^*$ the free extension of $\pi$.

If $\phi$ is a homomorphism of $(X, T, \pi)$ into another transformation group $(Y, T, \sigma)$, with $Y$ completely regular, evidently the map $\phi^*: A(X) \to A(Y)$ induced by $\phi$ is equivariant with respect to $\pi^*$ and $\sigma^*$.

**Theorem 1.** Suppose $T$ is locally compact. Then $\pi^*$ is continuous, $(A, T, \pi^*)$ is a transformation group whose transitions are automorphisms of $A$, and $j$ is an isomorphism of $(X, T, \pi)$ into $(A, T, \pi^*)$.

**Proof.** The only nontrivial fact to be proved is the continuity of $\pi^*$. Denote by $C(T, X)$, $C(T, A)$ the sets of all continuous maps of $T$ into $X$, $A$ respectively, and endow these function spaces with their compact-open topologies. Define $\mu^*: A \to A^T$ by $t(\varepsilon\mu^*) = (\varepsilon, t)\pi^*$ for $t \in T$, $\varepsilon \in A$. Since $T$ is locally compact, it suffices to show that $\mu^*$ maps $A$ continuously into $C(T, A)$. 
We show that \( A\mu^* \subset C(T, A) \). If \( x \in X \), then \( t \in T \) implies \( t(x\mu^*) = (x, t)\pi \), so \( x\mu^* \in C(T, A) \) by continuity of \( \pi \). Now let \( z \in A \). Choose \( x_1, \cdots, x_n \in X \) and integers \( \alpha_1, \cdots, \alpha_n \) with \( z = \sum \alpha_i x_i \). Then \( z\mu^* = \sum \alpha_i (x_i\mu^*) \), and \( z\mu^* \in C(T, A) \).

Let \( j^* \) be the canonical injection of \( C(T, X) \) into \( C(T, A) \), so that \( j^* \) is continuous. Define \( \mu: X \to X^T \) by \( t(x\mu) = (x, t)\pi \) for \( t \in T, x \in X \). Then \( \mu \) is a continuous map of \( X \) into \( C(T, X) \), so \( \psi = \mu j^* \) is a continuous map of \( X \) into \( C(T, A) \).

The addition on \( C(T, A) \) defined pointwise makes this space into a separated abelian group. Hence \( \psi \) extends to a continuous homomorphism \( \psi^* \) of \( A \) into \( C(T, A) \). But \( \mu^* \) is also a group homomorphism of \( A \) into \( C(T, A) \) extending \( \psi \). It follows that \( \mu^* = \psi^* \), and \( \mu^* \) is continuous.

**Corollary.** Suppose \( X \) is compact and \( (X, T, \pi) \) is equicontinuous. Then \( \pi^* \) is continuous.

**Proof.** Let \( E \) be the enveloping semigroup \([4]\) of \( (X, T, \pi) \). Then \( E \) is a compact group of homeomorphisms of \( X \) onto \( X \), and the evaluation map \( \sigma: X \times E \to X \) is continuous and defines a transformation group \( (X, E, \sigma) \). By the theorem the free extension \( \sigma^* \) of \( \sigma \) is continuous. The map \( \mu: T \to E \) such that \( t \in T \) implies \( t\mu = \pi^t \) is continuous. The continuity of \( \pi^* \) now follows from the factorization \( \pi^*(t)(x\mu) = (i^* \times \mu)(\sigma)^*(t) \), where \( i^* \) is the identity map of \( X \).

**Remark.** Instead of the free abelian topological group \( A \) over \( X \), consider the free linear topological space \( V \) over \( X \) \([9]\). If \( T \) is locally compact, then one can still show as above that \( \pi \) has a continuous extension \( \pi^*: V \times T \to V \) making \( (V, T, \pi^*) \) a transformation group of linear automorphisms of \( V \).

**4. Dynamical properties of** \((A(X), T)\). Again \((X, T, \pi)\) denotes a transformation group, where \( X \) is completely regular. Continuity of \( \pi^* \) is not needed below, and we suppress explicit mention of \( \pi \) and \( \pi^* \).

**Theorem 2.** The action of \( T \) on \( A \) is not topologically ergodic, that is, there exist nonempty open subsets \( N \) and \( M \) of \( A \) with \( Nt \cap M = \emptyset \) for all \( t \in T \).

**Proof.** Let \( D_0 \) be the constant map on \( X \) with range \( \{1\} \), let \( D \) be the homomorphism of \( A \) into the additive group of integers which extends \( D_0 \), and let \( N = \ker D \). Then \( N \) is a \( T \)-invariant subgroup of \( A \), so it is enough to show that \( N \) is open in \( A \).

Let \( \mathfrak{t}, \mathfrak{s} \) be the topologies of \( X, A \) respectively. The topology \( \mathfrak{s} \) of \( A \) generated by \( \{N\} \) and \( \mathfrak{t} \) is separated and compatible with the
Let \( s_0 \) be the topology on \( X \) induced by \( s \). It is now enough to show \( s_0 \subset \mathcal{J}_0 \), for then \( N \in s \subset \mathcal{J} \).

Let \( z \in A \) and \( V \in \mathcal{J} \) with \((z+N) \cap V \cap X \neq \emptyset \). It remains to show \((z+N) \cap V \cap X \subseteq \mathcal{J}_0 \). Choose any \( y \in (z+N) \cap X \). If \( x \in X \), then \( x = y + (x-y) \in z + N + N = z + N \). Hence \( X \subseteq z + N \), and \((z+N) \cap V \cap X = V \cap X \subseteq \mathcal{J}_0 \).

In case \( X \) is connected, the group \( N \) in the preceding proof is just the identity component of \( A \).

We use below a theorem of Ellis [3] stating that if \( T \) acts as a homeomorphism group of a space \( Y \) in which each orbit is relatively compact, then \((Y, T)\) is distal if and only if \((Y^n, T)\) is pointwise almost periodic for some \( n > 1 \), or equivalently, \((Y^n, T)\) is pointwise almost periodic for every \( n > 0 \). This applies to \((A, T)\) when \( X \) is compact, because \( A \) is the union of the compact sets \( A_n \) constructed in the proof of Proposition 1, and each \( A_n \) is \( T \)-invariant.

To accommodate the commutativity of addition in \( A \), it is convenient to use the symmetric product \( X \times X \) of \( X \) with itself [2]. Here \( X \times X \) is the compact space obtained by identifying each \((x, y) \in X \times X \) with \((y, x) \). Let \( p: X \times X \to X \times X \) be the projection, and for \((x, y) \in X \times X \) let \( x \cdot y \) denote \((x, y)p \). The map \((x \cdot y, t) \mapsto x \cdot t \cdot y \cdot t \) of \( X \times T \) into \( X \times X \) is well defined and makes \((X \times X, T)\) a transformation group and \( p \) a homomorphism.

**Theorem 3.** The following statements are equivalent when \( X \) is compact:

1. \((X, T)\) is distal.
2. \((A, T)\) is distal.
3. \((A, T)\) is pointwise almost periodic.

**Proof.** Assume (1). We show (2). Let \( z \in A \) with \( z \neq 0 \). It is enough to show that \( z \) is distal from 0. The compact \( T \)-invariant set \( A_1 = -X \cup X \cup \{0\} \) is distal under \( T \), so \((A^n_1, T)\) is distal for each \( n \). But for sufficiently large \( n \) both 0 and \( z \) belong to the range of the homomorphism \((\varepsilon_1, \ldots, \varepsilon_n) \mapsto \sum \varepsilon_i \) of \((A^n_1, T)\) into \((A, T)\).

By Ellis' theorem, (2) implies (3).

Assume (3). We show (1). We have an obvious isomorphism of \((X \times X, T)\) with \((X + X, T)\), so \((X \times X, T)\) is pointwise almost periodic. Let \( x, y \in X \) with \( x \neq y \). Choose disjoint compact neighborhoods \( U, V \) of \( x, y \) in \( X \). Then \((U \times V)p \) is a neighborhood of \( x \cdot y \) in \( X \times X \), and \((x \cdot y)T \subseteq (U \times V)pK \) for some compact subset \( K \) of \( T \). Then

\[
\alpha = (X \times X) \setminus ((U \times V) \cup (V \times U))K
\]
is an index of the uniformity of \( X \), and \((x, y)T\) is disjoint from \( \alpha \).

The following lemma concerning lifting of minimality is of interest in its own right (cf. [0, p. 27], [8, 2.1]).

**Lemma.** Let \((X, T), (Y, T)\) be transformation groups, where \( X, Y \) are compact, and let \( \phi \) be a locally one-to-one homomorphism of \((X, T)\) onto \((Y, T)\). Suppose \((Y, T)\) is minimal and \((X, T)\) has a dense orbit. Then \((X, T)\) is minimal.

**Proof.** Suppose \((X, T)\) is not minimal. Choose \( x_0 \in X \) with \( x_0 T \) dense in \( X \), and set \( y_0 = x_0 \phi \). There exists some minimal subset \( M \) of \( X \). The fiber \( y_0 \phi^{-1} \) over \( y_0 \) is finite and, since \( \phi \) maps \( M \) onto \( Y \), meets \( M \). Let \( y_0 \phi^{-1} = \{x_0, x_1, \ldots, x_n\} \) with \( y_0 \phi^{-1} \cap M = \{x_m, x_{m+1}, \ldots, x_n\} \).

Choose pairwise disjoint open neighborhoods \( W_0, \ldots, W_n \) of \( x_0, \ldots, x_n \) with \( \phi \) one-to-one on each \( W_i \) and with \( M \) disjoint from the closure of \( W_i \) for \( 0 < i < m \). For each \( z \in \bigcup_{0}^{n} W_i \) there exist disjoint neighborhoods of \( y_0 \phi^{-1} \) and \( z \phi^{-1} \) which are saturated by \( \phi \). A standard compactness argument produces a saturated neighborhood \( U \) of \( y_0 \phi^{-1} \) such that \( U \subseteq \bigcup_{0}^{n} W_i \). For \( i = 0, \ldots, n \) let \( U_i = U \cap W_i \), and let \( V = U \phi \), whence \( V \) is a neighborhood of \( y_0 \).

Because \((Y, T)\) is minimal it is discretely almost periodic at \( y_0 \), and there exist subsets \( S, K \) of \( T \) with \( K \) finite, \( T = SK \), and \( y_0 S \subseteq V \). Then \( x_i S \subseteq U \) for each \( i \).

There exists a net \((s_j, k_j)\) in \( S \times K \) such that \( \lim x_0 s_j k_j = x_m \). By passing to a subnet if necessary, we may assume \( k_j = k \) for some \( k \) and all \( j \). Then \( \lim x_0 s_j = x_m k^{-1} \in M \), so \( x_0 S \subseteq \bigcup_{0}^{n} U_i \), and \( x_0 S \subseteq U_p \) for some \( s \in S \) and some \( p \geq m \). It follows that some two of the \( n - m + 2 \) points \( x_0 s, x_m s, \ldots, x_n s \) belong to the same one of the \( n - m + 1 \) sets \( U_m, \ldots, U_n \). This is impossible since \( x_i s \phi = y_0 s \) for all \( i \) and \( \phi \) is one-to-one on each \( U_i \).

**Theorem 4.** Let \( x, y \in X \) with \( x \neq y \). Then \((x, y)\) is almost periodic under \((X \times X, T)\) if and only if \( x+y \) is almost periodic under \((A, T)\).

**Proof.** The restriction to \( X \times X \) of addition in \( A \) is a homomorphism of \((X \times X, T)\) into \((A, T)\). Hence \( x+y \) is almost periodic if \((x, y)\) is.

Conversely, assume \( x+y \) is almost periodic. We first show that \( x \) is distal from \( y \) in \( X \). Choose a symmetric index \( \alpha \) of the uniformity of \( X \) with \((x, y) \notin \alpha^3 \). Since \( x+y \) is almost periodic under \((X \times X, T)\), the set

\[
S = \{s \mid s \in T, (xs, ys) \in (x \alpha \times y \alpha) \cup (y \alpha \times x \alpha)\}
\]
is left syndetic in $T$, and $T = SK$ for some compact set $K$. Choose an index $\beta$ of $X$ with $\beta K^{-1} \subseteq \alpha$. Then $(x, y) T \cap \beta = \emptyset$.

Since $x$ is distal from $y$, the orbit-closure $B$ of $(x, y)$ in $X \times X$ is disjoint from the diagonal of $X \times X$. Then $\rho|B$ is a locally one-to-one (in fact, locally homeomorphic) homomorphism of $B$ onto the orbit-closure of $x*y$ in $X*X$. The almost periodicity of $(x, y)$ now follows from that of $x*y$ by means of the lemma.

A similar but more direct argument shows that $(x, y)$ is almost periodic if and only if $x - y$ is.

Added in proof. Mr. Leonard Shapiro has kindly pointed out that a proposition equivalent to our lemma appears in a paper of R. Ellis [Amer. J. Math 87 (1965), 564–574]. Ellis’ proof is entirely different from ours in that it employs the enveloping semigroup of a transformation group.

References


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