A SUMMABLE C_Ω-GROUP

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Recently, Megibben [3] has enriched, in a sense manyfold, the theory of torsion abelian groups by developing a "C_λ-theory" for each countable limit ordinal λ. The classical theory of torsion abelian groups corresponds to Megibben's C_ω-theory. Let p be a fixed prime. A p-primary group G belongs to the class C_λ, for any ordinal λ, if and only if $G/p^\alpha G$ is a direct sum of countable groups (d.s.c.) for each $\alpha<\lambda$. One should note that C_ω is the whole class of primary groups and that $C_\mu \subseteq C_\lambda$ is $\lambda \leqq \mu$. The question of to what extent the C_λ-theorems for a countable limit λ carry over to C_Ω-theorems, where Ω is the first uncountable ordinal, has been only partially settled. As we have already observed, $C_\Omega \subseteq C_\lambda$ but the point is C_Ω-theorems are stronger than C_λ-theorems for $\lambda<\Omega$. In particular, we have the following question. Can the countability condition on λ be removed in (either of) the next two theorems?

THEOREM (HILL AND MEGIBBEN [2]). *Let λ be a countable limit ordinal. If G is a summable C_λ-group of length λ, then G is a d.s.c.*

THEOREM (MEGIBBEN'S CRITERION). *Let λ be a countable limit ordinal. Let G be a summable, p-primary group of length λ. If, for each $\alpha<\lambda$, G contains a p^α-high subgroups which is a d.s.c., then G itself is a d.s.c.*

For the special case $\lambda=\omega$, Megibben's criterion is essentially the same as Kulikov's criterion, see [4]. But, of course, Megibben's theorem has much broader scope. The purpose of this paper is to show, however, that this scope does not encompass C_Ω-groups.

THEOREM. *A summable C_Ω-group need not be a d.s.c.*

PROOF. The proof is constructive. For each countable ordinal α let G_α be the unique countable, primary group of length α having each of its nonzero Ulm invariants equal to $\aleph_0$. Suppose that $\gamma<\Omega$ and that we have embedded G_α in G_β, for each $\alpha<\beta<\gamma$, in such a way that

$$(*) \qquad p^\lambda G_\beta \cap G_\alpha = p^\lambda G_\alpha \qquad \text{for all } \lambda$$

holds. If γ is a limit ordinal, then $G_\gamma \cong \bigcup_{\alpha<\gamma} G_\alpha$ and we have an embedding of G_α in G_γ such that (*) holds for $\alpha<\beta\leqq\gamma$. If γ is a nonlimit ordinal, we distinguish two cases.

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Case 1: $\gamma-1$ is a limit. Let $G_{\gamma-1}$ be a $p^{\gamma-1}$-high subgroup of G_γ.

Case 2. $\gamma-2$ exists. Let $G_{\gamma-1}$ be a direct summand of G_γ.

We remark, in connection with Case 1, that any $p^{\gamma-1}$-high subgroup of G_γ is isomorphic to $G_{\gamma-1}$. As far as Case 2 is concerned, $G_\gamma \cong G_{\gamma-1}+G_\gamma$. It is well known that $p^\lambda G_\gamma \cap G_{\gamma-1}=p^\lambda G_{\gamma-1}$ for either type of embedding described above, so condition (*) continues to hold for $\alpha<\beta\leqq\gamma$.

The group of interest is the union (=direct limit) of the groups G_α, as α ranges over the countable ordinals, embedded in one another in the precise manner described above. Let G denote this group, $G=\bigcup_{\alpha<\Omega} G_\alpha$. We wish to prove three things about G; we want to show that G is summable, that $G/p^\alpha G$ is a d.s.c for each countable α, and that G itself is not a d.s.c.

In order to show that G is summable, define $S_\alpha=p^\alpha G_{\alpha+1}[p]$ if α is a limit. If α is not a limit, let $G_{\alpha+1}=G_\alpha+K_\alpha$ and define $S_\alpha=K_\alpha[p]$. In either case, $G_{\alpha+1}[p]=G_\alpha[p]+S_\alpha$ is a *natural* decomposition [**2**] of $G_{\alpha+1}[p]$ in G—the decomposition honors heights computed in G. Thus $G[p]=\sum_{\alpha<\Omega} S_\alpha$ is a natural decomposition of all of $G[p]$. Since S_α is countable for each α, G is summable [**2**, Lemma 1.11].

In order to show that $G/p^\lambda G$ is a d.s.c. for each countable ordinal λ, we observe that the chain $\{G_\alpha, p^\lambda G\}/p^\lambda G$ of countable subgroups of $G/p^\lambda G$ leads up to $G/p^\lambda G$ and we analyze how these subgroups are embedded in one another. We claim that for $\alpha>\lambda$—λ is fixed—that any such subgroup splits out of the succeeding one, thereby giving a decomposition of $G/p^\lambda G$ into a direct sum of countable groups. In this connection, observe that $\{G_\beta, p^\lambda G\}/p^\lambda G=\bigcup_{\alpha<\beta}(\{G_\alpha, p^\lambda G\}/p^\lambda G)$ if β is a limit. Now we proceed to back up our claim. Suppose that $\alpha>\lambda$. There are, of course, two cases to consider in showing that $\{G_\alpha, p^\lambda G\}/p^\lambda G$ is a direct summand of $\{G_{\alpha+1}, p^\lambda G\}/p^\lambda G$.

Case 1. α is a limit ordinal. Recall that in this case G_α is a p^α-high subgroup of $G_{\alpha+1}$. Thus $\{G_\alpha, p^\lambda G_{\alpha+1}\}=G_{\alpha+1}$ since $\lambda<\alpha$, and $\{G_\alpha, p^\lambda G\}=\{G_{\alpha+1}, p^\lambda G\}$. In particular, $\{G_\alpha, p^\lambda G\}/p^\lambda G$ is a direct summand of $\{G_{\alpha+1}, p^\lambda G\}/p^\lambda G$; the two are equal.

Case 2. α is not a limit. In this case, $G_{\alpha+1}=G_\alpha+K_\alpha$. It follows at once that

$$\{G_{\alpha+1}, p^\lambda G\}/p^\lambda G = \{G_\alpha, p^\lambda G\}/p^\lambda G + \{K_\alpha, p^\lambda G\}/p^\lambda G.$$

This completes the argument that $G/p^\lambda G$ is a d.s.c. for each countable λ. Thus, by definition, G is a C_Ω-group.

Finally, we show that G is not a d.s.c. Assume that G is a direct sum of countable groups; let $G=\sum_{i\in I} A_i$ where A_i is countable for each $i\in I$. It is easy to show, by the back-and-forth method of [**2**], the existence of a countable subset J of I and a countable limit ordinal α such that $G_\alpha = \sum_{j\in J} A_j$. However, this is impossible since G_α is not a direct summand of G; G_α is not even a direct summand of $G_{\alpha+1}$ since G_α is p^α-high in $G_{\alpha+1}$. We conclude that G is not a d.s.c. and the theorem is proved.

COROLLARY. *Megibben's criterion does not apply to groups of length Ω.*

PROOF. According to Hill's theorem [1] that any isotype subgroup of a d.s.c. of countable length is a d.s.c., any summable C_Ω-group satisfies Megibben's criterion. Hence the above theorem yields the corollary.

REFERENCES

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