A NOTE ON INJECTIVE GROUP RINGS

ENZO R. GENTILE

In this note we shall prove the following result: *Let K be a commutative ring (with identity) and torsion free as a $\boldsymbol{Z}$-module. Then if G is any group then the group algebra $K(G)$ is left self-injective if and only if K is self-injective and G is finite.*

Part "if" was first proved by Eilenberg-Nakayama (see [**2**]). I. Connell (see [**4**, Appendix 2, Exercise **7**]) has proved that if $K(G)$ is left self-injective, for a commutative ring K, then K is self-injective and G is a locally finite group. So, the main result to be proved here is that if K is a torsion free commutative ring and $K(G)$ is left self-injective then G is a finite group.

In the present proof we shall need the following result from homological algebra (see [**1**, Chapter VI, Exercise 10]):

If $\theta\colon T\to S$ is a ring morphism and if S is (via θ) a flat T-module then any left S-module which is S-injective is T-injective.

Next we prove several partial results.

(a) *If H is a subgroup of G and $K(G)$ is left self-injective then $K(H)$ is a left self-injective ring.*

In fact, let $(g_j)_j$ be a system of representatives of left cosets of H in G, then $(g_j)_j$ is a base of $K(G)$ as a right $K(H)$-module, that is $K(G)$ is a right $K(H)$-free module. But by the remark above and from the fact that $K(G)$ is left self-injective we get that $K(G)$ is left $K(H)$-injective. Being $K(H)$ a direct summand (as $K(H)$-module) of $K(G)$ we conclude that $K(H)$ is left $K(H)$-injective. This proves our claim.

(b) *Let $\boldsymbol{Q}$ denote the rational field. Then if $\boldsymbol{Q}(G)$ is injective then G is a von Neumann ring.*

In fact, let $\boldsymbol{R}$ be the real field. Then it is a well-known result that $\boldsymbol{R}(G)$ is Jacobson semisimple (see [**6**]). Now clearly $\boldsymbol{R}(G)$ is right (and left) free $\boldsymbol{Q}(G)$-module. Since $\boldsymbol{R}(G)$ is $\boldsymbol{Q}(G)$-free and $\boldsymbol{R}(G)$ $\simeq \boldsymbol{Q}(G)\otimes_{\boldsymbol{Q}(G)}\boldsymbol{R}(G)$, Exercise 10, Chapter VI of [**1**] implies that $\boldsymbol{R}(G)$ is self-injective. Now Jacobson semisimple plus left self-injective implies von Neumann, according to a theorem by Utumi (see [**7**]). Let us recall the following result on von Neumann group rings (see [**8**, Theorem 8]): $\boldsymbol{Q}(G)$ is a von Neumann ring if (Auslander) and only if (Villamayor) G is locally finite. Therefore we have that G is locally finite. By using again the just mentioned theorem we get that $\boldsymbol{Q}(G)$ is a von Neumann ring as we wanted to prove.

Received by the editors April 1, 1969.

(c) *If $Q(G)$ is left self-injective then G is finite.*
According to (a) we can assume without loss of generality that G is a countable group. Therefore, in $Q(G)$, every left (or right) ideal is countably generated. Since $Q(G)$ is von Neumann a result by Kaplansky (see [**3**, Lemma 1]) shows that every left ideal is projective, or the same, $Q(G)$ is a left hereditary ring. Since $Q(G)$ is $Q(G)$-injective, every left cyclic $Q(G)$-module is injective. Now by a theorem of Osofsky (see [**5**]) $Q(G)$ is semisimple d.c.c. From this follows easily that G must be finite. In fact, the left ideal of $Q(G)$ generated by all $1-g$, g in G is a direct summand, so it is annihilated by an $x \in Q(G)$ $x \neq 0$. That is $x = g \cdot x$ for all $g \in Q(G)$. If we write $x = \sum_i a_i h_i$, a_i in Q, finite sum, we get that for every g in G, $h_i = h_j \cdot g$, that is, g belongs to the subgroup of G generated by the finite set (h_i). This shows that G is finite.

(d) *Let K be a torsion free commutative ring. If K is a self-injective ring then K contains the field Q, that is K is a Q-algebra.*
In fact, let $n \in N$. Then $nx \to x$ is a (well-defined) K-morphism of the ideal nK into K. So it can be realized by multiplication by an element a in K. This a is unique and is exactly n^{-1}. So clearly K contains the rationals.

Finally we are going to prove our result. Assume then that K is a torsion free ring and $K(G)$ left self-injective. First of all, notice that K is self-injective, because $K(G)$ is K-free, therefore $K(G)$ is K-injective and K is isomorphic to a K-direct summand of $K(G)$. By part (d) K contains the rationals. $K(G)$ is right free over $Q(G)$, therefore $Q(G)$ is left self-injective, so G is finite.

References

1. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., **1956**.

2. S. Eilenberg and T. Nakayama, *In the dimension of modules and algebras*. II, Nagoya Math. J. **9** (1955), 1–16.

3. I. Kaplansky, *On dimension of modules and algebras*. X, Nagoya Math. J. **13** (1958), 85–88.

4. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966.

5. B. Osofsky, *Rings all of whose finitely generated modules are injective*, Pacific J. Math. **14** (1964), 645–650.

6. N. Jacboson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

7. Y. Utumi, *On continuous rings and self-injective rings*, Trans. Amer. Math. Soc. **118** (1965), 158–173.

8. O. Villamayor, *On weak dimension of algebras*, Pacific J. Math. **9** (1959), 941–951.

Universidad Nacional de Buenos Aires