A NOTE ON INJECTIVE GROUP RINGS

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In this note we shall prove the following result: *Let K be a commutative ring* (*with identity*) *and torsion free as a* $\mathbf{Z}$*-module. Then if G is any group then the group algebra $K(G)$ is left self-injective if and only if K is self-injective and G is finite.*

Part "if" was first proved by Eilenberg-Nakayama (see [**2**]). I. Connell (see [**4**, Appendix 2, Exercise **7**]) has proved that if $K(G)$ is left self-injective, for a commutative ring K, then K is self-injective and G is a locally finite group. So, the main result to be proved here is that if K is a torsion free commutative ring and $K(G)$ is left self-injective then G is a finite group.

In the present proof we shall need the following result from homological algebra (see [**1**, Chapter VI, Exercise 10]):

If $\theta: T\to S$ is a ring morphism and if S is (via θ) a flat T-module then any left S-module which is S-injective is T-injective.

Next we prove several partial results.

(a) *If H is a subgroup of G and $K(G)$ is left self-injective then $K(H)$ is a left self-injective ring.*

In fact, let $(g_j)_j$ be a system of representatives of left cosets of H in G, then $(g_j)_j$ is a base of $K(G)$ as a right $K(H)$-module, that is $K(G)$ is a right $K(H)$-free module. But by the remark above and from the fact that $K(G)$ is left self-injective we get that $K(G)$ is left $K(H)$-injective. Being $K(H)$ a direct summand (as $K(H)$-module) of $K(G)$ we conclude that $K(H)$ is left $K(H)$-injective. This proves our claim.

(b) *Let $\mathbf{Q}$ denote the rational field. Then if $\mathbf{Q}(G)$ is injective then G is a von Neumann ring.*

In fact, let $\mathbf{R}$ be the real field. Then it is a well-known result that $\mathbf{R}(G)$ is Jacobson semisimple (see [**6**]). Now clearly $\mathbf{R}(G)$ is right (and left) free $\mathbf{Q}(G)$-module. Since $\mathbf{R}(G)$ is $\mathbf{Q}(G)$-free and $\mathbf{R}(G) \simeq \mathbf{Q}(G)\otimes_{\mathbf{Q}(G)}\mathbf{R}(G)$, Exercise 10, Chapter VI of [**1**] implies that $\mathbf{R}(G)$ is self-injective. Now Jacobson semisimple plus left self-injective implies von Neumann, according to a theorem by Utumi (see [**7**]). Let us recall the following result on von Neumann group rings (see [**8**, Theorem 8]): $\mathbf{Q}(G)$ is a von Neumann ring if (Auslander) and only if (Villamayor) G is locally finite. Therefore we have that G is locally finite. By using again the just mentioned theorem we get that $\mathbf{Q}(G)$ is a von Neumann ring as we wanted to prove.

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(c) *If $Q(G)$ is left self-injective then G is finite.*
According to (a) we can assume without loss of generality that G is a countable group. Therefore, in $Q(G)$, every left (or right) ideal is countably generated. Since $Q(G)$ is von Neumann a result by Kaplansky (see [**3**, Lemma 1]) shows that every left ideal is projective, or the same, $Q(G)$ is a left hereditary ring. Since $Q(G)$ is $Q(G)$-injective, every left cyclic $Q(G)$-module is injective. Now by a theorem of Osofsky (see [**5**]) $Q(G)$ is semisimple d.c.c. From this follows easily that G must be finite. In fact, the left ideal of $Q(G)$ generated by all $1-g$, g in G is a direct summand, so it is annihilated by an $x\in Q(G)$ $x\neq 0$. That is $x=g\cdot x$ for all $g\in Q(G)$. If we write $x=\sum_i a_i h_i$, a_i in Q, finite sum, we get that for every g in G, $h_i=h_j\cdot g$, that is, g belongs to the subgroup of G generated by the finite set (h_i). This shows that G is finite.

(d) *Let K be a torsion free commutative ring. If K is a self-injective ring then K contains the field Q, that is K is a Q-algebra.*
In fact, let $n\in N$. Then $nx\rightarrow x$ is a (well-defined) K-morphism of the ideal nK into K. So it can be realized by multiplication by an element a in K. This a is unique and is exactly n^{-1}. So clearly K contains the rationals.

Finally we are going to prove our result. Assume then that K is a torsion free ring and $K(G)$ left self-injective. First of all, notice that K is self-injective, because $K(G)$ is K-free, therefore $K(G)$ is K-injective and K is isomorphic to a K-direct summand of $K(G)$. By part (d) K contains the rationals. $K(G)$ is right free over $Q(G)$, therefore $Q(G)$ is left self-injective, so G is finite.

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