A GENERALIZED PALEY-WIENER-ZYGMUND INTEGRAL AND ITS APPLICATIONS

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1. **Introduction.** Let Y_N denote the product space $\prod_{j=1}^N [a_j, b_j]$ for $N=1, 2, \cdots$ and $p=(p_1, \cdots, p_N)$ any point in Y_N. The (N-dimensional) Yeh-Wiener space $C(Y_N)=C_N$ is the set of all real-valued continuous functions $f(p)$ on Y_N such that $f(p)=0$ if $p_j=a_j$ for some $1\leqq j\leqq N$. Thus C_N becomes the Wiener space C_W if $Y_N=[0, 1]$. (For the measure of this space consult with [**3**] and [**5**].) The definition of "bounded variation" and that of the ordinary R-S (Riemann-Stieltjes) integral in Y_N are found in [**5**].

It is known that if $h\in L^2(Y_N)$ and h_n the nth partial sum of the Fourier-series of h with respect to a C.O.N. set $\{\alpha_k\}$ with each α_k of B.V. on Y_N, then the P.W.Z. (Paley, Wiener, Zygmund) integral

$$\int_{Y_N} hd\overset{*}{f} \equiv \lim_n \int_{Y_N} h_n df, \quad \text{where} \int_{Y_N} h_n df$$

is the ordinary R-S integral, exists for almost all f in C_N, and that it is essentially independent of the particular choice of the C.O.N. set on Y_N, and also that the integral is essentially consistent with the ordinary R-S integral if h is of B.V. on Y_N. The case when $N=1$ was discussed in [**1**], for $N=2$ in [**2**], and for more general cases in [**3**].

Some basic questions that arise here are these:

If $h\in L^2(Y_N)$, does there exist

$$\int_{Y_N} hfd\overset{*}{f} \tag{1.1}$$

for almost all f in C_N? If it does, is it essentially consistent for various C.O.N. sets? The answer to these questions is much deeper than it

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appears to be. Thus, if the above quoted properties for the P.W.Z. integral hold for all f instead of almost all, the answer would be affirmative, since $h\in L^2(Y_N)$ and $f\in C_N$ imply $hf\in L^2(Y_N)$. However, since the P.W.Z. integral does not exist for all f in C_N, it implies nothing about (1.1). It is easy to see that $\int_{Y_N} hgd^*f$ must exist for almost all (f, g) in $C_N\times C_N$. But the set where $f=g$ is a null set in $C_N\times C_N$, and thus the almost everywhere existence of $\int_{Y_N}hgd^*f$ does not imply that (1.1) exists anywhere.

Using other techniques, however, we shall show that for a large class of C.O.N. sets, the existence of (1.1) for almost all f in C_N can be proved, and the equality of the values of (1.1) for any two C.O.N. sets of the class holds for almost all f in C_N. Existence theorem but no consistency theorem was obtained by Colson [4] for the corresponding problem for functions of one variable under somewhat similar condition on C.O.N. sets.

2. Statement of main results.

THEOREM 1. *Let $h\in L^2(Y_N)$ and let $\{\alpha_k\}$ be a real-valued C.O.N. set of functions on Y_N each of which is of B.V. on Y_N. Furthermore let*

$$\lim_n \sum_{k=1}^{n} \alpha_k(p) \int_{A(p)} \alpha_k(q)d\mu_N(q) = 2^{-N} \tag{2.1}$$

in L^2-sense on Y_N, where $A(p)=\prod_{j=1}^{N} [a_j, p_j]$ for each $p=(p_1, \cdots, p_N)$ in Y_N and μ_N the Lebesgue measure in R_N. For each f in C_N let $(hf)_n$ denote the nth partial sum of the Fourier series of hf with respect to $\{\alpha_k\}$. Then l.i.m.$_n\int_{Y_N}(hf)_n df$ *exists in L^2-sense on C_N.*

THEOREM 2. *If h, $\{\alpha_k\}$, and $(hf)_n$ are as in Theorem* 1, *then*

$$\int_{Y_N} hf d^*f \equiv \lim_n \int_{Y_N} (hf)_n df$$

exists for almost all f in C_N.

THEOREM 3. *If $h\in L^2(Y_N)$, then $\int_{Y_N}hfd^*f$ is essentially consistent with different choices of C.O.N. sets in the sense that if $\{\alpha_k\}$ and $\{\beta_k\}$ are two different C.O.N. sets satisfying the conditions of Theorem* 1, *then*

$$\{\alpha\}\int_{Y_N} hfd^*f = \{\beta\}\int_{Y_N} hfd^*f$$

for almost all f in C_N.

COROLLARY. *If $h\in L^2(Y_N)$, then*

$$E_N^f\left[\int_{Y_N} hf d^* f\right]^2 = \frac{1}{4}\int_{Y_N} h^2(p)\prod_{j=1}^{N}(p_j - a_j)d\mu_N(p) + 4^{-(N+1)}\left(\int_{Y_N} h d\mu_N\right)^2,$$

where $E_N^f[F(f)]$ denotes the integral $\int_{C_N}F(f)dm_N(f)$, m_N being the Yeh-Wiener measure in C_N, and $p=(p_1, \cdots, p_N)$, $\prod_{j=1}^N (p_j-a_j) \equiv (p_1-a_1)\cdots(p_N-a_N)$.

THEOREM 4. *If $h\in L^2(Y_N)$, f is absolutely continuous on Y_N, and if $\partial^N f/\partial p_1\cdots\partial p_N\in L^2(Y_N)$, then the P.W.Z. integral $\int_{Y_N} hd^*f$ converges and is equal to the Lebesgue integral $\int_{Y_N} h\cdot(\partial^N f/\partial p_1\cdots\partial p_N)d\mu_N$.*

COROLLARY. *If h and f are as in the theorem, then*

$$\int_{Y_N} hf^n d^* f = \int_{Y_N} hf^n(\partial^N f/\partial p_1, \cdots, \partial p_N)d\mu_N \quad \text{for } n = 0, 1, 2, \cdots.$$

THEOREM 5. *Let $h(t)$ be absolutely continuous and $h'(t)$ of class $L^2[0, 1]$. Then $\int_0^1 hfd^*f=1/2\int_0^1 h(t)d[f^2(t)]$ for almost all $f\in C_W$.*

THEOREM 6. *Let $\{\alpha_k(t)\}$ and $\{\beta_k(t)\}$ be the C.O.N. cosine and sine functions on $[0, 1]$ respectively, i.e., $\alpha_0(t)=1$, $\alpha_k(t)=2^{1/2}\cos k\pi t$, $k=1, 2, \cdots$, and $\beta_k(t)=2^{1/2}\sin k\pi t$, $k=1, 2, \cdots$. For each $f\in C_W$ let $a_k=a_k(f)=\int_0^1 f(t)\alpha_k(t)dt$ and $b_k=b_k(f)=\int_0^1 f(t)\beta_k(t)dt$. Then for almost all f in C_W we have*

(i) $\sum_{k=1}^\infty ka_kb_k=-f^2(1)/(2\pi)$,

(ii) $\sum_{k=1}^\infty (-1)^k a_k=2^{-1/2}[f(1)-\int_0^1 f(t)dt]$,

(iii) $\sum_{k=1}^\infty a_{2k}=(2^{1/2}/4)[f(1)-2\int_0^1 f(t)dt]$,

(iv) $\sum_{k=1}^\infty a_k=(-2^{1/2}/2)\int_0^1 f(t)dt$.

COROLLARY. *The Fourier cosine series converges to zero at the origin for almost all f in C_W.*

REMARKS. 1. The convergence of (i), (ii), (iii), and (iv) in the theorem holds true for every absolutely continuous functions f on $[0, 1]$ with $f'\in L^2[0, 1]$ by virtue of the corollary to Theorem 4 with $N=1$.

2. The corollary is a good contrast to DuBois-Reymond's Theorem (see [9]) which asserts the existence of a continuous function whose Fourier series does not converge at the origin.

3. **Complete orthonormal sets satisfying our conditions.** (I) Let $\{\alpha_k(x)\}$ be the C.O.N. cosine functions on $[a, b]$, i.e.,

$$\alpha_1(x) = \frac{1}{\sqrt{b-a}}, \quad \alpha_k(x) = \sqrt{\frac{2}{b-a}} \cos \frac{(k-1)\pi(x-a)}{b-a}, \qquad k = 2, 3, \cdots.$$

Then

$$\lim_n \sum_{k=1}^{n} \alpha_k(x) \int_a^x \alpha_k(s)ds = \frac{1}{2} \tag{3.1}$$

in L^2-sense on $[a, b]$.

A straightforward computation leads to

$$\lim_n \int_a^b \left[\sum_{k=1}^{n} \alpha_k(x) \int_a^x \alpha_k(s)ds - \frac{1}{2}\right]^2 dx = 0.$$

Direct computations also show that the C.O.N. sine functions, the C.O.N. trigonometric functions, and the C.O.N. set obtained from Legendre polynomials satisfy (3.1). More generally we give the following, whose proof is given in [2].

(II) Let $\{u_k(x)\}$ be the C.O.N. characteristic functions of the Sturm-Liouville system

$$\begin{aligned} &[p(x)u'(x)]' + q(x)\rho - r(x)]u(x) = 0 \\ &u'(a) - \lambda \cdot u(a) = 0, \quad u'(b) + \mu \cdot u(b) = 0 \end{aligned} \tag{3.2}$$

where $p(x)$, $q(x)$, and $r(x)$ are continuous and positive on $[a, b]$, and let each $u_k(x)$ correspond to the characteristic number ρ_k with $\rho_k \uparrow$. Then it follows that

$$\lim_n \sum_{k=1}^{n} u_k(x) \int_a^x u_k(t)dt = \frac{1}{2} \quad \text{on } [a, b]. \tag{3.3}$$

(III) The C.O.N. sets obtained above are directly connected with the satisfactory C.O.N. sets for the definition of $\int_{Y_N} hfd^*f$ as follows:

For each j, $1 \leq j \leq N$, let $\{\alpha_k^{(j)}\}$ be a C.O.N. set of functions satisfying (3.3) on $[a_j, b_j]$. Then

$$\lim_n \sum_{k_1,\cdots,k_N=1}^{n} \alpha_{k_1}^{(1)}(p_1)\cdots\alpha_{k_N}^{(N)}(p_N) \cdot \int_{A(p)} \alpha_{k_1}^{(1)}(q_1)\cdots\alpha_{k_N}^{(N)}(q_N)d\mu_N(q) = 2^{-N}.$$

This follows immediately by observing that

$$\sum_{k_1,\cdots,k_N=1}^{n} \alpha_{k_1}^{(1)}(p_1)\cdots\alpha_{k_N}^{(N)}(p_N)\int_{A(p)} \alpha_{k_1}^{(1)}(q_1)\cdots\alpha_{k_N}^{(N)}(q_N)d\mu_N(q) = \prod_{j=1}^{N}\left[\sum_{k_j=1}^{n}\alpha_{k_j}^{(j)}(p_j)\int_{a_j}^{p_j}\alpha_{k_j}^{(j)}(q_j)dq_j\right].$$

4. **Necessary lemmas.** For the proof of the theorems we need the following lemmas.

LEMMA 1 (P.W.Z. THEOREM). *Let α_j, $j=1, 2, \cdots, n$, be an orthonormal set each of which is of B.V. on Y_N. Let $F(u_1, \cdots, u_n)$ be a Lebesgue-measurable function on R_n. Then*

$$E_N^f\left[F\left(\int_{Y_N}\alpha_1 df, \cdots, \int_{Y_N}\alpha_n df\right)\right] = \pi^{-n/2}\int_{R_n}F(u_1, \cdots, u_n)\exp\left(-\sum_{j=1}^{n}u_j^2\right)du_1\cdots du_n$$

in the sense the existence of one side implies the existence of the other and the equality. (*See* [6].)

LEMMA 2. *If h_j, $j=1, 2, 3, 4$, are functions of B.V. on Y_N, then*

$$E_N^f\left[\prod_{j=1}^{4}\int_{Y_N}h_j df\right] = \frac{1}{4}\left[\int_{Y_N}h_1h_2d\mu_N\cdot\int_{Y_N}h_3h_4d\mu_N + \int_{Y_N}h_1h_3d\mu_N\cdot\int_{Y_N}h_2h_4d\mu_N + \int_{Y_N}h_1h_4d\mu_N\cdot\int_{Y_N}h_2h_3d\mu_N\right].$$

This lemma can be established by othonormalizing the h_j and then using Lemma 1.

LEMMA 3 (BEARMAN'S THEOREM). *Let $F(f, g)$ be an integrable functional on $C_N \times C_N$. Then*

$$E_N^g(E_N^f[F(f, g)]) = E_N^g(E_N^f[F(f\cos\theta - g\sin\theta, f\sin\theta + g\cos\theta)]).$$

Bearman [**7**] has proved this theorem for $N=1$. The validity of this theorem for arbitrary N is immediate.

5. **Proof of Theorem 1.** By the definition of $(hf)_n$ we have

$$\int_{Y_N} (hf)_n df = \sum_{k=1}^{n}\left(\int_{Y_N} hf\alpha_k d\mu_N\right)\int_{Y_N}\alpha_k df,$$

and for $n>m$

$$\begin{aligned} I_{m,n} &= E_N^f\left[\int_{Y_N}(hf)_n df - \int_{Y_N}(hf)_m df\right]^2 \\ &= \sum_{j,k=m}^{n} E_N^f\left[\int_{Y_N} hf\alpha_j d\mu_N\cdot\int_{Y_N} hf\alpha_k d\mu_N\cdot\int_{Y_N}\alpha_j df\cdot\int_{Y_N}\alpha_k df\right] \\ &= \sum_{j,k=m}^{n} E_N^f\left[\int_{Y_N^2} h(p)f(p)\alpha_j(p)h(q)f(q)\alpha_k(q)\left(\int_{Y_N}\alpha_j df\right)\right. \\ &\qquad\left.\cdot\left(\int_{Y_N}\alpha_k df\right) d\mu_{2N}\right]. \end{aligned} \tag{5.1}$$

By Fubini's theorem

$$I_{m,n} = \sum_{j,k=m}^{n}\int_{Y_N^2} h(p)\alpha_j(p)h(q)\alpha_k(q) \cdot\left\{E_N^f\left[f(p)f(q)\left(\int_{Y_N}\alpha_j df\right)\left(\int_{Y_N}\alpha_k df\right)\right]\right\} d\mu_{2N}. \tag{5.2}$$

Since $f\in C_N$ and $p, q\in Y_N$, we may write $f(p)=\int_{a_N}^{p_N}\cdots\int_{a_1}^{p_1} df=\int_{A(p)} df$ and $f(q)=\int_{A(q)} df$, where $A(p)=\prod_{j=1}^{N}[a_j, p_j]$. We define a function $\xi(p, s)$ on Y_N^2 by

$$\begin{aligned} \xi(p, s) &= 1 \quad \text{if } s_i \leqq p_i \quad \text{for } i = 1, \cdots, N \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{5.3}$$

Then

$$f(p) = \int_{Y_N} \xi(p, s)df(s), \qquad f(q) = \int_{Y_N} \xi(q, s)df(s). \tag{5.4}$$

Using these expressions and then applying Lemma 3, we obtain

$$\begin{aligned}
E_N^f&\left[f(p)f(q)\left(\int_{Y_N} \alpha_j df\right)\left(\int_{Y_N} \alpha_k df\right)\right] \\
&= \frac{1}{4}\left[\int_{Y_N} \xi(p, s)\xi(q, s)d\mu_N(s) \cdot \int_{Y_N} \alpha_j\alpha_k d\mu_N\right. \\
&\quad + \int_{Y_N} \xi(p, s)\alpha_j(s)d\mu_N(s) \cdot \int_{Y_N} \xi(q, s)\alpha_k(s)d\mu_N(s) \\
&\quad \left.+ \int_{Y_N} \xi(p, s)\alpha_k(s)d\mu_N(s) \cdot \int_{Y_N} \xi(q, s)\alpha_j(s)d\mu_N(s)\right] \\
&= \frac{1}{4}\left\{\delta_{jk} \cdot \prod_{i=1}^{N} [\min(p_i, q_i) - a_i] + \int_{A(p)} \alpha_j d\mu_N \cdot \int_{A(q)} \alpha_k d\mu_N\right. \\
&\quad \left.+ \int_{A(p)} \alpha_k d\mu_N \cdot \int_{A(q)} \alpha_j d\mu_N\right\}.
\end{aligned}$$

Use this result in (5.2) and then rearrange terms to get

$$\begin{aligned}
4I_{m,n} &= \sum_{k=m}^{n} \int_{Y_N^2} h(p)\alpha_k(p)h(q)\alpha_k(q) \prod_{i=1}^{N} [\min(p_i, q_i) - a_i]d\mu_{2N} \\
&\quad + \int_{Y_N} h(p)\left[\sum_{j=m}^{n} \alpha_j(p) \int_{A(p)} \alpha_j(s)d\mu_N(s)\right]d\mu_N(p) \\
&\quad \cdot \int_{Y_N} h(q)\left[\sum_{k=m}^{n} \alpha_k(q) \int_{A(q)} \alpha_k(s)d\mu_N(s)\right]d\mu_N(q) \\
&\quad + \sum_{j,k=m}^{n} \int_{Y_N^2} h(p)\alpha_j(p)h(q)\alpha_k(q)\left(\int_{A(p)} \alpha_k d\mu_N\right) \\
&\quad \cdot\left(\int_{A(q)} \alpha_j d\mu_N\right) d\mu_{2N}.
\end{aligned} \tag{5.5}$$

If we use "integration by parts" repeatedly to eliminate each factor $[\min(p_i, q_i) - a_i]$, we obtain

$$\begin{aligned}\int_{Y_N^2} h(p)\alpha_k(p)h(q)\alpha_k(q) \prod_{i=1}^{N} [\min(p_i, q_i) - a_i]d\mu_{2N} \\ = \int_{Y_N}\left[\int_{B(q)} h(p)\alpha_k(p)d\mu_N(p)\right]^2 d\mu_N(q) \\ = \int_{Y_N}\left[\int_{Y_N} h(p)\alpha_k(p)\chi_{B(q)}(p)d\mu_N(p)\right]^2 d\mu_N(q),\end{aligned}$$

where $B(q)=\prod_{i=1}^{N} [q_i, b_i]$ and $\chi_{B(q)}(p)$ the characteristic function of $B(q)$. Hence on applying monotone convergence theorem and Parseval's equation, we get

$$\begin{aligned}\sum_{k=1}^{\infty}\int_{Y_N^2} h(p)\alpha_k(p)h(q)\alpha_k(q) \prod_{i=1}^{N} [\min(p_i, q_i) - a_i]d\mu_{2N} \\ = \int_{Y_N}\left\{\int_{Y_N} [h(p)\chi_{B(q)}(p)]^2 d\mu_N(p)\right\} d\mu_N(q) \\ = \int_{Y_N}\left\{\int_{B(q)} h^2(p)d\mu_N(p)\right\} d\mu_N(q) \\ = \int_{Y_N} h^2(p)\prod_{i=1}^{N} (p_i - a_i)d\mu_N(p).\end{aligned} \tag{5.6}$$

Using the function ξ defined by (5.3), we may write

$$\begin{aligned}\sum_{j,k=1}^{\infty} \int_{Y_N^2} h(p)\alpha_j(p)h(q)\alpha_k(q)\left(\int_{A(p)}\alpha_k d\mu_N\right)\left(\int_{A(q)}\alpha_j d\mu_N\right) d\mu_{2N} \\ = \sum_{j,k=1}^{\infty} \int_{Y_N^2} h(p)\xi(p, s)\alpha_j(p)\alpha_k(s)d\mu_{2N} \\ \cdot\int_{Y_N^2} h(q)\xi(q, s)\alpha_k(q)\alpha_j(s)d\mu_{2N} \\ = \int_{Y_N^2} h(p)\xi(p, s)h(s)\xi(s, p)d\mu_{2N} = 0,\end{aligned} \tag{5.7}$$

where the expression, second from the last, follows by Parseval's equation, and the last follows from the fact that $\xi(p, s)\cdot\xi(s, p)=0$ a.e. on Y_N^2. Use (2.1), (5.6) and (5.7) in (5.5) to conclude that $\lim_{m,n} I_{m,n}=0$, thus proving the theorem. Furthermore, if we take $m=0$ in $I_{m,n}$,

$$\lim_n E_N^f\left[\int_{Y_N}(hf)_n df\right]^2$$

$$(5.8)\qquad =\lim_n I_{0,n}$$

$$=\frac{1}{4}\left[\int_{Y_N} h^2(p)\prod_{i=1}^N (p_i - a_i)d\mu_N(p) + 4^{-N}\left(\int_{Y_N} hd\mu_N\right)^2\right].$$

6. **Proof of Theorem 2.** Let S be the set of all $(f, g)\in C_N\times C_N$ for which $\int_{Y_N} hgd^*f$ exists. Then, since $h\in L^2$ and $g\in C_N$ imply $hg\in L^2$, it follows that the integral $\int_{Y_N} hgd^*f$ exists for almost all $(f, g)\in C_N\times C_N$, i.e., $E_N^g\{E_N^f[\chi_S(f, g)]\}=1$, where χ_S is the characteristic function of S. Hence by Lemma 3, it follows that

$$(6.1)\qquad E_N^g\{E_N^f[\chi_S(f\cos\theta - g\sin\theta, f\sin\theta + g\cos\theta)]\} = 1.$$

By taking $\theta=\pi/4$ in (6.1), we obtain

$$E_N^g\left\{E_N^f\left[\chi_S\left(\frac{f-g}{\sqrt{2}}, \frac{f+g}{\sqrt{2}}\right)\right]\right\} = 1,$$

from which it follows that

$$\int_{Y_N} h\cdot\frac{f-g}{\sqrt{2}}\, d^*\left(\frac{f+g}{\sqrt{2}}\right)$$

exists for a.a. $(f, g)\in C_N\times C_N$. Using the definition of the integral we may rewrite this in the form

$$(6.2)\qquad \lim_n \frac{1}{2}\int_{Y_N}[h\cdot(f-g)]_n d(f+g) \text{ exists for a.a. } (f, g)\in C_N\times C_N,$$

where $[h\cdot(f-g)]_n$ is as in Theorem 1. Similarly $\theta=-\pi/4$ yields

$$(6.3)\qquad \lim_n \frac{1}{2}\int_{Y_N}[h\cdot(f+g)]_n d(-f+g) \text{ exists for a.a. } (f, g)\in C_N\times C_N.$$

Subtracting (6.3) from (6.2), we find that

$$(6.4)\qquad \lim_n\left[\int_{Y_N}(hf)_n df - \int_{Y_N}(hg)_n dg\right] \text{ exists for a.a. } (f, g)\in C_N\times C_N.$$

Hence we can choose a fixed $g_0\in C_N$ such that

$$\lim_n \left[\int_{Y_N} (hf)_n df - \int_{Y_N} (hg_0)_n dg_0 \right] \text{ exists for a.a. } f \in C_N. \tag{6.5}$$

Thus, if we show that $\lim_n \int_{Y_N} (hg_0)_n dg_0$ exists, then $\lim_n \int_{Y_N} (hf)_n df$ will exist for a.a. $f \in C_N$ and hence the proof will complete.

To show the existence of $\lim_n \int_{Y_N} (hg_0)_n dg_0$, we assume the opposite. Then there exist a positive ϵ and $\{(m_i, n_i)\}$ of positive integers, with $m_i, n_i \to \infty$ as $i \to \infty$, such that

$$\int_{Y_N} (hg_0)_{n_i} dg_0 - \int_{Y_N} (hg_0)_{m_i} dg_0 \geqq \epsilon, \qquad i = 1, 2, \cdots. \tag{6.6}$$

Now,

$$\begin{aligned} &\liminf_i \left[\int_{Y_N} (hf)_{n_i} df - \int_{Y_N} (hf)_{m_i} df \right] \\ &\geqq \liminf_i \left\{ \left[\int_{Y_N} (hf)_{n_i} df - \int_{Y_N} (hg_0)_{n_i} dg_0 \right] \right. \\ &\qquad \left. - \left[\int_{Y_N} (hf)_{m_i} df - \int_{Y_N} (hg_0)_{m_i} dg_0 \right] \right\} \\ &\quad + \liminf_i \left[\int_{Y_N} (hg_0)_{n_i} dg_0 - \int_{Y_N} (hg_0)_{m_i} dg_0 \right]. \end{aligned} \tag{6.7}$$

Use (6.5) and (6.6) in (6.7) to obtain

$$\liminf_i \left[\int_{Y_N} (hf)_{n_i} df - \int_{Y_N} (hf)_{m_i} df \right] \geqq \epsilon \qquad \text{for a.a. } f \in C_N. \tag{6.8}$$

On the other hand by Theorem 1, $\lim_n \int_{Y_N} (hf)_n df$ exists in L^2-sense on C_N, thus giving

$$\lim_i E_N^f \left[\int_{Y_N} (hf)_{n_i} df - \int_{Y_N} (hf)_{m_i} df \right]^2 = 0. \tag{6.9}$$

On using Fatou's lemma in (6.9), we see that

$$\liminf_i \left[\int_{Y_N} (hf)_{n_i} df - \int_{Y_N} (hf)_{m_i} df \right]^2 = 0 \qquad \text{for a.a. } f \in C_N.$$

This is a contradiction to (6.8), thus completing the proof.

The corollary is a simple result of Theorems 1 and 2 together with (5.8). The proof of Theorem 3 is rather lengthy and can not be included in this paper. However, the proof for the case $N=2$ is available in [2], and general cases follow in similar manner.

7. **Proof of Theorem 4.** Let $\{\alpha_k\}$ be a C.O.N. set for the definition of $\int_{Y_N} hd^*f$. Let $e_k=\int_{Y_N} h\alpha_k d\mu_N$, $k=1, 2, \cdots$. Then by the definition of P.W.Z. integral we have

$$\int_{Y_N} hd^*f = \lim_n \int_{Y_N} \left(\sum_{k=1}^n e_k\alpha_k\right) df$$
$$= \sum_{k=1}^\infty \left(\int_{Y_N} h\alpha_k d\mu_N\right)\left(\int_{Y_N} \alpha_k df\right).$$

But $\int_{Y_N}\alpha_k df$ is an ordinary Riemann-Stieltjes integral, and by the assumption on f we have $\int_{Y_N}\alpha_k df=\int_{Y_N}\alpha_k\cdot(\partial^N f/\partial p_1\cdots\partial p_N)d\mu_N$. (This identity may be found on p. 668 in [8].) Upon using Parseval's equation, we obtain

$$\int_{Y_N} hd^*f = \sum_{k=1}^\infty \left(\int_{Y_N} h\alpha_k d\mu_N\right)\left[\int_{Y_N} \alpha_k\cdot(\partial^N f/\partial p_1\cdots\partial p_N)d\mu_N\right]$$
$$= \int_{Y_N} h\cdot(\partial^N f/\partial p_1\cdots\partial p_N)d\mu_N,$$

which was to be shown.

The proof of Theorem 5 is similar to that of Theorem 1.

8. **Proof of Theorem 6.** (i) Choose $h=1$ in Theorem 5 to get

$$\int_0^1 fd^*f = \frac{1}{2}\int_0^1 d[f^2(t)] = \frac{1}{2}f^2(1) \qquad \text{for a.a. } f\in C_W. \tag{8.1}$$

On the other hand, if we use the definition directly with $\{\beta_k\} = \{2^{1/2}\sin k\pi t\}$, it follows that

$$\begin{aligned}\int_0^1 fd^*f &= \sum_{k=1}^\infty b_k\int_0^1 2^{1/2}\sin k\pi t df(t)\\ &= \sum_{k=1}^\infty b_k\left[-\int_0^1 f(t)k\pi 2^{1/2}\cos k\pi t dt\right] \qquad (8.2)\\ &= -\pi\sum_{k=1}^\infty k a_k b_k.\end{aligned}$$

Hence (i) follows from (8.1) and (8.2).

(ii) By the use of C.O.N. cosine functions $\{1, 2^{1/2}\cos k\pi t\}$ for the definition of P.W.Z. integral, we find that

$$
\begin{aligned}
\int_0^1 fd^*f &= \int_0^1 f(t)dt \cdot \int_0^1 df(t) + \sum_{k=1}^{\infty} a_k \int_0^1 2^{1/2}\cos k\pi t df(t) \\
&= f(1)\int_0^1 f(t)dt \\
&\quad + \sum_{k=1}^{\infty} a_k \left[f(1)2^{1/2}\cos k\pi + k\pi \int_0^1 f(t)2^{1/2}\sin k\pi t dt \right] \\
&= f(1)\int_0^1 f(t)dt + 2^{1/2}f(1)\sum_{k=1}^{\infty}(-1)^k a_k + \pi \sum_{k=1}^{\infty} k a_k b_k.
\end{aligned} \tag{8.3}
$$

Substituting (i) in (8.3) and then using the fact that $f(1)=0$ for only a set of functions of Wiener measure zero in C_W, we finally obtain the result (ii).

(iii) We now use the C.O.N. trigonometric functions $\{1, 2^{1/2}\sin 2k\pi t, 2^{1/2}\cos 2k\pi t\}$ for $\int_0^1 fd^*f$ to get

$$
\begin{aligned}
\int_0^1 fd^*f = \int_0^1 f(t)dt \cdot \int_0^1 df(t) \\
+ \sum_{k=1}^{\infty}\Bigg[b_{2k} \int_0^1 2^{1/2}\sin 2k\pi t df(t) \\
+ a_{2k}\int_0^1 2^{1/2}\cos 2k\pi t df(t) \Bigg].
\end{aligned} \tag{8.4}
$$

After using "integration by parts" for the R-S integral on the right member of (8.4), we find that

$$
\int_0^1 fd^*f = f(1)\int_0^1 f(t)dt + 2^{1/2}f(1)\sum_{k=1}^{\infty} a_{2k}. \tag{8.5}
$$

Substitution of (8.1) in (8.5) yields (iii).

(iv) Now, from (ii) and (iii)

$$
\sum_{k=1}^{\infty} a_k = 2\sum_{k=1}^{\infty} a_{2k} - \sum_{k=1}^{\infty}(-1)^k a_k = -2^{-1/2}\int_0^1 f(t)dt \quad \text{for a.a. } f \in C_W.
$$

The corollary is a simple result of (iv).

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