ON A THEOREM OF POKORNYI

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Let $p_0, p_1, \cdots, p_{n-1}$ be analytic functions defined in a region R. The differential equation

$$(1) \qquad y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = 0$$

is said to be disconjugate in R if no nontrivial solution of (1) has more than $n-1$ zeros (where the zeros are counted with their multiplicities) in R. For the even-order equation ($n=2m$), we may consider a weaker notion of disconjugacy: Equation (1) is said to be disconjugate in the sense of Reid [9] in R if no nontrivial solution of (1) has two zeros of order m in R.

Disconjugacy of the second-order equation

$$(2) \qquad y'' + py = 0$$

has been studied by Nehari [5], [7], Pokornyi [8], and London [4]; the results are usually formulated as univalence criteria for an analytic function. In [8], Pokornyi announced the following theorem: Let p be analytic in $D=\{z: |z|<1\}$. If

$$|p(z)| \leqq 2/(1-|z|^2), \qquad z \in D,$$

then Equation (2) is disconjugate in D.

The principal aim of this note is to establish an analogous result for the equation

$$(3) \qquad y^{(2m)} + py = 0.$$

THEOREM I. *Let p be analytic in $D=\{z: |z|<1\}$. If*

$$|p(z)| \leqq (2m)!/(1-|z|^2)^m, \qquad z \in D,$$

then Equation (3) *is disconjugate in the sense of Reid in D.*

For the proof of the above theorem, we require the following lemma.

LEMMA I. *Let y be analytic in a region R. If $y(a_i)=0$, $a_i \in R$, $i=1, 2, \cdots, n$,* then

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$$y(z_0) = (a_n - z_0) \cdots (a_2 - z_0) \int_{a_1}^{z_0} \frac{1}{(a_2 - z_1)^2} \int_{a_2}^{z_1} \frac{a_2 - z_2}{(a_3 - z_2)^3} \cdots$$

$$\cdot \int_{a_{n-1}}^{z_{n-2}} \frac{(a_{n-1} - z_{n-1})^{n-2}}{(a_n - z_{n-1})^n} \int_{a_n}^{z_{n-1}} (a_n - z_n)^{n-1} y^{(n)}(z_n) dz_n \cdots dz_1,$$

where the integrations are performed along any curve in R *connecting the two points* a_k *and* z_{k-1}, $k = 1, 2, \cdots, n$ (*cf.* [**1**]).

PROOF. If f is analytic in R and $f(a) = 0$, $a \in R$, then it is easily confirmed that

$$\left(\frac{f}{a - z}\right)^{(k)} = \frac{1}{(a - z)^{k+1}} \int_a^z (a - w)^k f^{(k+1)}(w) dw, \tag{4}$$

$k = 0, 1, 2, \cdots$. The lemma now follows from (4) and induction.

PROOF OF THEOREM I. Suppose that Equation (3) has a nontrivial solution y with two zeros $z = a_1$ and $z = a_2$ of order m in D. Choose constants K and α, $|K| = 1$, $|\alpha| < 1$, such that the transformation $z = K(w - \alpha)/(1 - \bar{\alpha} w)$ carries $z = a_1$ and $z = a_2$ onto $w = 0$ and $w = -\rho$, $0 < \rho < 1$, respectively. Then the function Y defined by

$$Y(w) = y\left(\frac{K(w - \alpha)}{1 - \bar{\alpha} w}\right) \cdot \exp\left[-(2m - 1) \int \frac{\bar{\alpha}}{1 - \bar{\alpha} w} dw\right]$$

has two zeros $w = 0$ and $w = -\rho$ of order m and satisfies the differential equation

$$Y^{(2m)} + \left[\frac{K(1 - |\alpha|^2)}{(1 - \bar{\alpha} w)^2}\right]^{2m} qY = 0, \tag{5}$$

where $q(w) = p(K(w - \alpha)/(1 - \bar{\alpha} w))$ (see, e.g., [**3**]). Furthermore, we have

$$\left|\frac{K(1 - |\alpha|^2)}{(1 - \bar{\alpha} w)^2}\right|^{2m} |q(w)| \leqq \frac{(2m)!}{(1 - |w|^2)^m} \tag{6}$$

for $-1 < w \leqq 0$. Since the transformation $z = K(w - \alpha)/(1 - \bar{\alpha} w)$ can be built up from two rotations and a transformation of the type $z = (w - \beta)/(1 - \beta w)$, $0 < \beta < 1$, it suffices to establish (6) for these two types of transformations. That (6) holds for $z = Kw$, $|K| = 1$, is readily seen. For $z = (w - \beta)/(1 - \beta w)$, $0 < \beta < 1$, we have

$$\left|\frac{1-\beta^2}{(1-\beta w)^2}\right|^{2m}\left|p\left(\frac{w-\beta}{1-\beta w}\right)\right|$$

$$\leqq \left|\frac{1-\beta^2}{(1-\beta w)^2}\right|^{2m} \frac{(2m)!}{(1-|(w-\beta)/(1-\beta w)|^2)^m}$$

$$= \frac{(2m)!}{(1-|w|^2)^m}\left|\frac{1-\beta^2}{(1-\beta w)^2}\right|^m$$

$$\leqq \frac{(2m)!}{(1-|w|^2)^m}, \qquad -1 < w \leqq 0.$$

We now use Lemma I to express the function Y in the interval $[-\rho, 0]$:

$$Y(w) = -(\rho+w)^{m-1}w^m \int_{-\rho}^{w} \frac{1}{(\rho+w_1)^2}\int_{-\rho}^{w_1}\frac{1}{(\rho+w_2)^2}\cdots$$

$$\tag{7} \cdot\int_{-\rho}^{w_{m-2}} \frac{1}{(\rho+w_{m-1})^2}\int_{-\rho}^{w_{m-1}} \frac{(\rho+w_m)^{m-1}}{w_m^{m+1}}\int_0^{w_m}\frac{1}{w_{m+1}^2}\cdots$$

$$\cdot\int_0^{w_{2m-1}}\frac{1}{w_{2m-1}^2}\int_0^{w_{2m-1}} w_{2m}^{2m-1}Y^{(2m)}(w_{2m})dw_{2m}\cdots dw_1,$$

where the integrations are performed along the negative real axis. Since $|Y^{(2m)}(w)|$ is a continuous function defined on the compact interval $[-\rho, 0]$, it attains its maximum at some point $w=w_0$, $-\rho \leqq w_0 \leqq 0$. Taking the absolute values and integrating (7), we arrive at

$$\tag{8} |Y(w)| \leqq |Y^{(2m)}(w_0)|\,|w|^m|\rho+w|^m/(2m)!, \qquad -\rho \leqq w \leqq 0.$$

Finally, from (5) and (8), we deduce

$$|Y^{(2m)}(w)| \leqq \frac{1}{(2m)!}|Y^{(2m)}(w_0)|\left|\frac{K(1-|\alpha|^2)}{(1-\bar{\alpha}w)^2}\right|^{2m}|q(w)|\,|w|^m|\rho+w|^m$$

for $-\rho \leqq w \leqq 0$; in particular, for $w=w_0$,

$$1 \leqq \frac{1}{(2m)!}\left|\frac{K(1-|\alpha|^2)}{(1-\bar{\alpha}w_0)^2}\right|^{2m}|q(w_0)|\,|w_0|^m|\rho+w_0|^m$$

$$< \frac{1}{(2m)!}\left|\frac{K(1-|\alpha|^2)}{(1-\bar{\alpha}w_0)^2}\right|^{2m}|q(w_0)|(1-|w_0|^2)^m,$$

contrary to (6). This contradiction proves the theorem.

This theorem for the case $m=2$ was previously obtained by Hadass [2].

Disconjugacy criteria of a somewhat different nature may be obtained with the help of the following inequalities [4], [6]: If p is analytic in $D=\{z: |z|<1\}$, $z=x+iy$, then

$$|p(w)| \leqq \frac{\int_0^{2\pi} |p(e^{i\theta})|\, d\theta}{2\pi(1-|w|^2)}, \qquad w \in D,$$

and

$$|p(w)| \leqq \frac{\iint_{|z|<1} |p(z)|\, dxdy}{\pi(1-|w|^2)^2}, \qquad w \in D.$$

From these inequalities and Theorem I results the following theorem.

THEOREM II. *Let p be analytic in $D=\{z: |z|<1\}$. If*

$$\int_0^{2\pi} |p(e^{i\theta})|\, d\theta \leqq 2\pi(2m)!,$$

or if $m \geqq 2$ and if

$$\iint_{|z|<1} |p(z)|\, dxdy \leqq \pi(2m)!,$$

then Equation (3) is disconjugate in the sense of Reid in D.

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