ON A THEOREM OF POKORNyi

W. J. KIM

Let $p_0, p_1, \ldots, p_{n-1}$ be analytic functions defined in a region $R$. The differential equation

$$(1) \quad y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0y = 0$$

is said to be disconjugate in $R$ if no nontrivial solution of (1) has more than $n-1$ zeros (where the zeros are counted with their multiplicities) in $R$. For the even-order equation ($n = 2m$), we may consider a weaker notion of disconjugacy: Equation (1) is said to be disconjugate in the sense of Reid [9] in $R$ if no nontrivial solution of (1) has two zeros of order $m$ in $R$.

Disconjugacy of the second-order equation

$$(2) \quad y'' + py = 0$$

has been studied by Nehari [5], [7], Pokornyi [8], and London [4]; the results are usually formulated as univalence criteria for an analytic function. In [8], Pokornyi announced the following theorem: Let $p$ be analytic in $D = \{z : |z| < 1\}$. If

$$|p(z)| \leq 2/(1 - |z|^2), \quad z \in D,$$

then Equation (2) is disconjugate in $D$.

The principal aim of this note is to establish an analogous result for the equation

$$(3) \quad y^{(2m)} + py = 0.$$ 

**Theorem I.** Let $p$ be analytic in $D = \{z : |z| < 1\}$. If

$$|p(z)| \leq (2m)!/(1 - |z|^2)^m, \quad z \in D,$$

then Equation (3) is disconjugate in the sense of Reid in $D$.

For the proof of the above theorem, we require the following lemma.

**Lemma I.** Let $y$ be analytic in a region $R$. If $y(a_i) = 0$, $a_i \in R$, $i = 1, 2, \ldots, n$, then

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where the integrations are performed along any curve in \( R \) connecting the two points \( a_k \) and \( z_{k-1} \), \( k = 1, 2, \ldots, n \) (cf. [1]).

PROOF. If \( f \) is analytic in \( R \) and \( f(a) = 0 \), \( a \in R \), then it is easily confirmed that

\[
\left( \frac{f}{a - z} \right)^{(k)} = \frac{1}{(a - z)^{k+1}} \int_a^z (a - w)^k f^{(k+1)}(w)dw,
\]

\( k = 0, 1, 2, \ldots \). The lemma now follows from (4) and induction.

PROOF OF THEOREM I. Suppose that Equation (3) has a nontrivial solution \( y \) with two zeros \( z = a_1 \) and \( z = a_2 \) of order \( m \) in \( D \). Choose constants \( K \) and \( \alpha, |K| = 1, |\alpha| < 1 \), such that the transformation

\( z = K(w - \alpha)/(1 - \alpha w) \)

carries \( z = a_1 \) and \( z = a_2 \) onto \( w = 0 \) and \( w = -\rho, 0 < \rho < 1 \), respectively. Then the function \( Y \) defined by

\[
Y(w) = y \left( \frac{K(w - \alpha)}{1 - \alpha w} \right) \cdot \exp \left[ -(2m - 1) \int \frac{\bar{\alpha}}{1 - \alpha w} dw \right]
\]

has two zeros \( w = 0 \) and \( w = -\rho \) of order \( m \) and satisfies the differential equation

\[
Y^{(2m)} + \left[ \frac{K(1 - |\alpha|^2)}{(1 - \alpha w)^2} \right]^{2m} qY = 0,
\]

where \( q(w) = p(K(w - \alpha)/(1 - \alpha w)) \) (see, e.g., [3]). Furthermore, we have

\[
\left| \frac{K(1 - |\alpha|^2)}{(1 - \alpha w)^2} \right|^{2m} |q(w)| \leq \frac{(2m)!}{(1 - |w|^2)^m}
\]

for \(-1 < w \leq 0\). Since the transformation \( z = K(w - \alpha)/(1 - \alpha w) \) can be built up from two rotations and a transformation of the type \( z = (w - \beta)/(1 - \beta w), 0 < \beta < 1 \), it suffices to establish (6) for these two types of transformations. That (6) holds for \( z = Kw, |K| = 1 \), is readily seen. For \( z = (w - \beta)/(1 - \beta w), 0 < \beta < 1 \), we have
We now use Lemma I to express the function $Y$ in the interval $[-\rho, 0]$:

$$Y(w) = - (\rho + w)^{m-1} w^m \int_{-\rho}^{w} \frac{1}{(\rho + w_1)^2} \int_{-\rho}^{w_1} \frac{1}{(\rho + w_2)^2} \cdots$$

$$\cdot \int_{-\rho}^{w_{m-2}} \frac{1}{(\rho + w_{m-1})^2} \int_{-\rho}^{w_{m-1}} \frac{1}{(\rho + w_m)^{m-1}} \int_{0}^{w_m} \frac{1}{w_{m+1}} \cdots \cdot d\omega_{2m-1} d\omega_{2m-2} \cdots d\omega_{1}.$$  \hspace{1cm} (7)

where the integrations are performed along the negative real axis. Since $|Y^{(2m)}(w)|$ is a continuous function defined on the compact interval $[-\rho, 0]$, it attains its maximum at some point $w = w_0$, $-\rho \leq w_0 \leq 0$. Taking the absolute values and integrating (7), we arrive at

$$|Y(w)| \leq |Y^{(2m)}(w_0)| |w|^m |\rho + w|^m/(2m)!, \quad -\rho \leq w \leq 0.$$ \hspace{1cm} (8)

Finally, from (5) and (8), we deduce

$$|Y^{(2m)}(w)| \leq \frac{1}{(2m)!} |Y^{(2m)}(w_0)| \left| \frac{K(1 - |\alpha|^2)}{(1 - \alpha \omega)^2} \right|^{2m} |q(w)| |w|^m |\rho + w|^m$$

for $-\rho \leq w \leq 0$; in particular, for $w = w_0$,

$$1 \leq \frac{1}{(2m)!} \left| \frac{K(1 - |\alpha|^2)}{(1 - \alpha \omega_0^2)} \right|^{2m} |q(w_0)| |w_0|^m |\rho + w_0|^m$$

$$< \frac{1}{(2m)!} \left| \frac{K(1 - |\alpha|^2)}{(1 - \alpha \omega_0^2)} \right|^{2m} |q(w_0)| (1 - |w_0|^2)^m,$$

contrary to (6). This contradiction proves the theorem.

This theorem for the case $m=2$ was previously obtained by Hadass [2].
Disconjugacy criteria of a somewhat different nature may be obtained with the help of the following inequalities [4], [6]: If \( p \) is analytic in \( D = \{ z : |z| < 1 \} \), \( z = x + iy \), then

\[
| p(w) | \leq \frac{\int_0^{2\pi} | p(e^{i\theta}) | d\theta}{2\pi(1 - |w|^2)}, \quad w \in D,
\]

and

\[
| p(w) | \leq \frac{\iint_{|z|<1} | p(z) | \, dx \, dy}{\pi(1 - |w|^2)^2}, \quad w \in D.
\]

From these inequalities and Theorem I results the following theorem.

**Theorem II.** Let \( p \) be analytic in \( D = \{ z : |z| < 1 \} \). If

\[
\int_0^{2\pi} | p(e^{i\theta}) | \, d\theta \leq 2\pi(2m)!,
\]

or if \( m \geq 2 \) and if

\[
\iint_{|z|<1} | p(z) | \, dx \, dy \leq \pi(2m)!,
\]

then Equation (3) is disconjugate in the sense of Reid in \( D \).

**References**


