BOUNDED SOLUTIONS OF SOME ABSTRACT DIFFERENTIAL EQUATIONS

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Introduction. We give here two theorems abstract versions and generalization of some results in our previous papers [**2**], [**3**].[2] For semigroup theory applied here look at [**1**].

THEOREM 1. *Let $T(t)$ be a strongly continuous semigroup in the Banach space X such that $T(o)=I$ and* $\lim_{t\to\infty} T(t)x=0$ $\forall x\in X$.

Let A be its infinitesimal generator and suppose $\exists\lambda_0\in \mathbf{C}$ such that $(\lambda_0-A)^{-1}$ is a compact operator in X.

Let $u(t)\in C^1(-\infty,\infty;X)$, $u(t)\in D(A)$ $\forall t\in(-\infty,\infty)$, be a solution of the abstract differential equation

$$u'(t) = Au(t)$$

which is defined for all $t\in(-\infty,\infty)$.

Suppose $\exists c>0$ such that $\int_t^{t+1}\|u(t)\|_X^2dt\leqq c$, $\forall t\in(-\infty,\infty)$. Then $u(t)\equiv\theta$, $\forall t\in(-\infty,\infty)$.

PROOF OF THE THEOREM. We remark first that from our hypothesis

$$\sup_{-\infty<t<\infty}\int_t^{t+1}\|u(t)\|_X^2dt<\infty$$

there follows the existence of a sequence $(t_n)_1^\infty\downarrow-\infty$, such that $\sup_{n\in N}\|u(t_n)\|_X<\infty$. We remark also that, for each $t\geqq t_0$, the function $u(t)$ admits the representation $u(t)=T(t-t_0)u(t_0)$.

Also, a well-known fact is that $(\lambda_0-A)^{-1}T(t)=T(t)(\lambda_0-A)^{-1}$. Let us put now: $\omega(t)=(\lambda_0-A)^{-1}u(t)$; obviously sequence $\{\omega(t_n)\}_1^\infty$ contains a convergent subsequence $\omega(t_{n_k})\to\omega_\infty\in X$, and $\omega(t)=T(t-t_0)\omega(t_0)$, $\forall t\geqq t_0$. Let $t\in(-\infty,\infty)$ be fixed and k_0 big enough to get $t_{n_k}<t$ for $k\geqq k_0$. Then we obtain, as $t>t_{n_k}$,

$$\omega(t) = T(t-t_{n_k})\omega(t_{n_k}) = T(t-t_{n_k})[\omega(t_{n_k})-\omega_\infty] + T(t-t_{n_k})\omega_\infty$$

and, consequently,

Received by the editors February 17, 1969.

[1] Supported by the N.R.C. of Canada.

[2] We thank the referee whose report helped us to improve the presentation of Theorem 1 and the original Theorem 2.

$$\begin{aligned}\|\omega(t)\| &\leqq \|T(t-t_{n_k})\|_{\mathcal{L}(X,X)}\|\omega(t_{n_k})-\omega_\infty\|_X + \|T(t-t_{n_k})\omega_\infty\|_X \\ &\leqq L\|\omega(t_{n_k})-\omega_\infty\|_X + \|T(t-t_{n_k})\omega_\infty\|_X, \qquad \forall k \geqq k_0.\end{aligned}$$

(Because of the strong continuity of $T(t)$ and asymptotic decay to 0, it follows that $\|T(t)\|_{\mathcal{L}} \leqq L$.) This obviously implies $\omega(t)=\theta$, hence $u(t)=\theta$.

In the second result, conditions on the compactness of $(\lambda_0-A)^{-1}$ and on strong asymptotic decay of $T(t)$ are replaced by the condition of strong asymptotic decay of the adjoint semigroup $T^*(t)$. Precisely, we have

THEOREM 2. *Let $T(t)$ be a strongly continuous semigroup in the Banach space X, and let us suppose that*

$$\lim_{t\to\infty} T^*(t)x^* = 0, \qquad \forall x^* \in X^*$$

$T^*(t)$ being, $\forall t \in (0, \infty)$, the adjoint operator of $T(t)$ acting in the dual space X^*. Let A be the infinitesimal generator of $T(t)$ and $u(t)$; $-\infty < t < \infty \Rightarrow D(A)$ be a strong solution of the differential equation $u'(t)=Au(t)$. Suppose $\exists c>0$ such that

$$\int_t^{t+1} \|u(t)\|_X^2 dt \leqq c, \qquad -\infty < t < \infty.$$

Then $u(t)\equiv\theta$, $-\infty<t<\infty$.

PROOF OF THEOREM 2. As in Theorem 1 we find a sequence $(t_n)_1^\infty$ of real numbers, such that $\lim_{n\to\infty} t_n = -\infty$ and $\|u(t_n)\|_X \leqq M$, $n=1, 2, \cdots$. For arbitrary real t_0, we will have $t_n<t_0$ if $n\geqq n_0$. Then we may represent $u(t_0)$ as: $u(t_0)=T(t_0-t_n)u(t_n)$. Take now an arbitrary $x^*\in X^*$; then, if $\langle\ ,\ \rangle$ means duality between X and X^* we will have $\langle x^*, u(t_0)\rangle = \langle x^*, T(t_0-t_n)u(t_n)\rangle = \langle T^*(t_0-t_n)x^*, u(t_n)\rangle$ and

$$|\langle x^*, u(t_0)\rangle| \leqq \|T^*(t_0-t_n)x^*\|\,\|u(t_n)\| \leqq M\|T^*(t_0-t_n)x^*\|.$$

As $n\to\infty$, $T^*(t_0-t_n)x^*\to\theta$; hence $\langle x^*, u(t_0)\rangle=\theta\ \forall x^*\in X^*$ and consequently $u(t_0)=\theta$.

REMARK. An example of the situation arising in Theorem 2 is the following: (see our paper [3]).

Take $X=L^2(R^n)$; $T(t)f=e^{-(s_1^2+\cdots+s_n^2)t}f(s)_{t\geqq 0}$, $s=(s_1\cdots s_n)$, $f(s)\in L^2(R^n)$. It is immediate that $T^*(t)=T(t)$, $\forall t\geqq 0$, and that

$$\lim_{t\to\infty}\left(\int_{R^n} e^{-2(s_1^2+\cdots+s_n^2)t}f^2(s)ds\right)^{1/2} = 0, \qquad \forall f(s)\in L^2.$$

Also, $Af=-(s_1^2+\cdots+s_n^2)f(s)$ with $D(A)=\{f\in L^2,\ -|s|^2f\in L^2\}$. Hence our Theorem 2 applies in this case.

References

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