

A NOTE ON THE REPRESENTATION OF A SOLUTION OF AN ELLIPTIC DIFFERENTIAL EQUATION NEAR AN ISOLATED SINGULARITY¹

DAVID G. SCHAEFFER²

There are a number of results known which state that a solution u of an elliptic differential equation

$$(1) \quad Au = 0$$

which has an isolated singularity at a point $p \in \mathbf{R}^n$ may be expressed as the sum of a derivative of the fundamental solution of A and a solution of (1) regular at p , providing that u satisfies one of various conditions limiting its growth near p (see for example F. John [2] or R. Seeley [7]). The main conclusion of this note is a representation of any solution of (1) with an isolated singularity at p which makes no assumption on the behavior of u near the singularity; the representation is in terms of a (real) analytic functional supported on $\{p\}$ applied to the fundamental solution. This result is in the spirit of the work of J. L. Lions and E. Magenes [3] on elliptic boundary value problems with analytic functionals as data.

Actually with our method it involves no additional difficulty to obtain the representation when u is singular on a compact set $K \subset \mathbf{R}^n$ —that is, when u is a solution of (1) on $\Omega \sim K$, where Ω is some open connected neighborhood of K in \mathbf{R}^n . We may suppose without loss of generality that $\partial\Omega$ is smooth and that u is \mathcal{C}^∞ on $\bar{\Omega} \sim K$, because any neighborhood of K contains a smaller neighborhood for which this will be true. We assume that A is a properly elliptic differential operator (as defined by M. Schechter in [6]) of order $2m$ whose coefficients are analytic on $\bar{\Omega}$. Let γ be a two-sided fundamental solution for A on Ω ; more explicitly, if $\Gamma: \mathcal{D}(\Omega) \rightarrow \mathcal{E}(\Omega)$ is defined by

$$\Gamma\phi(x) = \int_{\Omega} dx' \gamma(x, x')\phi(x'),$$

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then

$$(2) \quad \Gamma A\phi = A\Gamma\phi = \phi$$

for all $\phi \in \mathcal{D}(\Omega)$. The existence of such a fundamental solution was proved by B. Malgrange in [4].

To specify the notation we review in this paragraph the terminology of analytic functionals (see A. Martineau [5] for details). We define the space of functions $\mathcal{A}(V)$ analytic on an open set V in \mathbb{R}^n as the inductive limit as $\epsilon \rightarrow 0$ of the space of functions on V whose power series about any point converges in a ball of radius ϵ . That is, if $\epsilon > 0$, let

$$\|\psi\|_\epsilon = \sup_{x \in V} \sup_{\alpha} (\alpha!)^{-1} \epsilon^{|\alpha|} |D^\alpha \psi(x)|,$$

where α is a multi-index exponent for the differentiation operator D ; let

$$\mathcal{A}(V; \epsilon) = \{ \psi \in \mathcal{E}(V) \mid \|\psi\|_\epsilon < \infty \}$$

and give it the norm topology; and finally let

$$\mathcal{A}(V) = \text{ind lim}_{\epsilon \rightarrow 0} \mathcal{A}(V; \epsilon).$$

In the usual way, the space of functions $\mathcal{A}(K)$ analytic on a closed set K is defined as the inductive limit of the space of functions analytic on some neighborhood V of K as V decreases to K . A (real) analytic functional supported on K is a continuous linear functional on $\mathcal{A}(K)$, an element of the dual space $\mathcal{A}'(K)$.

We remark that the fundamental solution for A is analytic, because A has analytic coefficients. Thus if $x \in \Omega \sim K$, then $D_x^\alpha \gamma(x, \cdot) \in \mathcal{A}(K)$ for any multi-index α , and the difference quotients for these derivatives converge in the topology of $\mathcal{A}(K)$. In particular,

$$A\gamma(x, \cdot) = 0 \in \mathcal{A}(K)$$

for $x \in \Omega \sim K$. If $T \in \mathcal{A}'(K)$, we denote by $T[\gamma]$ the function

$$(3) \quad v(x) = T[\gamma(x, \cdot)] \quad (x \in \Omega \sim K).$$

It is readily shown by an exchange of limits that $Av = 0$. We state now our main theorem.

THEOREM 1. *If u is a solution of (1) on $\Omega \sim K$, then there is an analytic functional T supported on K such that $u - T[\gamma]$ is the restriction to $\Omega \sim K$ of an (analytic) solution of (1) defined on Ω .*

Before we prove the theorem we introduce a space of solutions on K of the adjoint equation and we construct from u a certain linear functional on this space which characterizes the singularity of u on K .³ Let

$$\mathcal{J}(K) = \{\psi \in \mathcal{A}(K) \mid A^*\psi = 0\},$$

and give it the relative topology. If ϕ is a smooth function on $\Omega \sim K$ such that $A\phi \in L^1(\Omega \sim K)$ and if $\psi \in \mathcal{J}(V)$ [that is, the kernel of A^* in $\mathcal{A}(V)$, where V is some neighborhood of K], let ρ be a C^∞ function supported in V that is identically one near K and define

$$(4) \quad B[\phi, \psi] = \int_{\Omega \sim K} dx \{ \phi A^*(\rho\psi) - \rho\psi A\phi \}.$$

Since $A^*(\rho\psi)$ has compact support, the possibly troublesome first term of the integral in (4) is well defined. The integral is independent of the choice of ρ because the difference of two possible choices is a test function supported in $\Omega \sim K$, permitting an integration by parts. For each V the functional $B[\phi, \cdot]$ is continuous on $\mathcal{J}(V)$, so $B[\phi, \cdot] \in \mathcal{J}'(K)$ by inductive limits. The functional $B[u, \cdot]$ specifies the boundary data of u on ∂K in the sense of equation (6) below.

Suppose w is a smooth function on $\bar{\Omega}$ such that A^*w vanishes in a neighborhood V of K ; choose ρ as in (4) and let $\zeta = 1 - \rho$, so that ζ is a C^∞ function vanishing near K . Consider the integral

$$(5) \quad \int_{\Omega \sim K} dx \{ u A^*w - w A u \} = \int_{\Omega \sim K} dx \{ u A^*(\rho w) - \rho w A u \} + \int_{\Omega \sim K} dx \{ u A^*(\zeta w) - \zeta w A u \};$$

the first term on the right in (5) is simply $B[u, w]$, while the second reduces to a surface integral by Green's theorem. Hence

$$(6) \quad \int_{\Omega \sim K} dx \{ u A^*w - w A u \} = B[u, w] + \sum_{j=0}^{2m-1} \int_{\partial\Omega} d\sigma \left(\frac{\partial}{\partial\nu} \right)^j u D_j w,$$

where $\partial/\partial\nu$ denotes the exterior normal derivative and D_j is a differential operator of order $2m - j - 1$ for which $\partial\Omega$ is noncharacteristic.

PROPOSITION 2. *If u is a solution of $Au = 0$ on $\Omega \sim K$, then u is the restriction to $\Omega \sim K$ of a solution on Ω if and only if $B[u, \cdot] = 0$.*

³ This procedure is very suggestive of the duality considered by A. Grothendieck in [1] and by others.

PROOF. If u extends to a solution of (1) on Ω , then an integration by parts in (4) checks that $B[u, \cdot] = 0$. Conversely, suppose that $B[u, \cdot] = 0$; we show that an extension of u to Ω may be obtained as a solution u' of the Dirichlet problem, $Au' = 0$ in Ω , whose Dirichlet data on $\partial\Omega$ coincides with that of u . A solution u' exists, for if w is any solution of the adjoint equation $A^*w = 0$ with homogeneous data, then by (6)

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} d\sigma \left(\frac{\partial}{\partial\nu}\right)^j u D_j w = 0;$$

that is, the data is orthogonal to any solution of the adjoint equation, so a solution exists according to the Fredholm alternative.

Let N denote the finite-dimensional space of solutions of (1) on Ω with vanishing Dirichlet data. If $f \in \mathfrak{D}(\Omega \sim K)$ is orthogonal to N , choose w so that $A^*w = f$ and w has homogeneous data. Then again by (6)

$$\int_{\Omega \sim K} dx u' A^*w = \sum_{j=0}^{m-1} \int_{\partial\Omega} d\sigma \left(\frac{\partial}{\partial\nu}\right)^j u' D_j w = \int_{\Omega \sim K} dx u A^*w,$$

thus

$$\int dx (u - u') f = 0.$$

Hence $(u - u')$ is orthogonal to any vector in N^\perp , so u differs from u' by an element of N which may be added to u' to obtain the desired extension.

PROOF OF THEOREM 1. Suppose u is a solution of (1) on $\Omega \sim K$. For $x \in \Omega \sim K$ we define the function

$$v(x) = B[u, \gamma(x, \cdot)].$$

By the Hahn-Banach theorem $B[u, \cdot]$ may be extended from $\mathfrak{g}(K)$ to a linear functional T on $\mathfrak{Q}(K)$, so v is of the form $T[\gamma]$. As we remarked before Theorem 1, $Av = 0$; we show below that $B[v, \cdot] = B[u, \cdot]$. Hence by the proposition $u - v$ is the restriction to $\Omega \sim K$ of a solution of (1) on Ω .

If $w \in \mathfrak{g}(K)$,

$$\begin{aligned} B[v, w] &= \int dx v(x) A^*[\rho w(x)] = \int dx T[\gamma(x, \cdot)] A^*[\rho w(x)] \\ &= T \left\{ \int dx \gamma(x, \cdot) A^*[\rho w(x)] \right\}, \end{aligned}$$

since the fact that $A^*(\rho w)$ has compact support in $\Omega \sim K$ implies that the integral converges in the topology of $\mathcal{G}(K)$. Thus

$$(7) \quad B[v, w] = T[\Gamma^* A^*(\rho w)],$$

where $\Gamma^* \phi(x) = \int dx' \bar{\gamma}(x', x) \phi(x')$. It is obvious from (2) that $\Gamma^* A^* \phi = A^* \Gamma^* \phi = \phi$ for all $\phi \in \mathcal{D}(\Omega)$; moreover, since $\rho \equiv 1$ near K , $\Gamma^* A^*(\rho w) = w$ near K . Therefore from (7), $B[v, w] = T[w] = B[u, w]$, where the final equality follows from the observation that $w \in \mathcal{G}(K)$. This completes the proof.

We remark in closing that a similar representation for a solution of the inhomogeneous equation $Au = f$ can be proved quite simply with our methods.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY