ON THE CONVOLUTION OF LOGARITHMICALLY
CONCAVE SEQUENCES

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In [1], Davenport and Polya have considered the following problem. If \( \sum p_r x^r \) and \( \sum q_r x^r \) are two series with positive coefficients and if

\[
( \sum p_r x^r)( \sum q_r x^r) = \sum W_r x^r
\]

then what conditions will ensure that the coefficients \( W_r \) shall be logarithmically convex? We say that \( W_r \) is logarithmically convex if

\[
(W_r)^2 \leq W_{r-1} W_{r+1}, \quad r = 1, 2, 3, \ldots.
\]

If

\[
p_r = p_r / \alpha_r,
\]

\[
q_r = q_r / \beta_r,
\]

\[
\alpha_r = \frac{\alpha(\alpha + 1) \cdots (\alpha + r - 1)}{1 \cdot 2 \cdot 3 \cdots r},
\]

\[
\beta_r = \frac{\beta(\beta + 1) \cdots (\beta + r - 1)}{1 \cdot 2 \cdot 3 \cdots r},
\]

\( \alpha > 0, \beta > 0, \alpha + \beta = 1 \) and if \( p_r \) and \( q_r \) are both logarithmically convex then Davenport and Polya have proved in [1], that \( W_r \) is logarithmically convex, where

\[
W_r = \alpha_0 p_0 \beta_0 q_r + \alpha_1 p_1 \beta_{r-1} q_{r-1} + \cdots + \alpha_r p_r \beta_0 q_0.
\]

It must be noted that the result of Davenport and Polya is false with the omission of the weights \( \alpha_r \) and \( \beta_r \) as defined in (5) and (6) respectively. In this paper we prove a similar result for logarithmically concave sequences.

**Definition.** A sequence \( \{ \alpha_r \} \) is said to be logarithmically concave if

\[
\alpha_r^2 \geq \alpha_{r-1} \alpha_{r+1}, \quad (r = 1, 2, 3, \ldots).
\]

Evidently a positive sequence is logarithmically concave if and only if it satisfies the relations

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\[
\alpha_1/\alpha_0 \geq \alpha_2/\alpha_1 \geq \alpha_3/\alpha_2 \geq \cdots.
\]

**Theorem.** Let \( \{p_r\} \) and \( \{q_r\} \) be positive logarithmically concave sequences with \( p_0 = q_0 = 1 \). Then the sequence \( \{W_r\} \) is also logarithmically concave, where the \( W_r \) are defined by the product of formal power series

\[
\sum_{r=0}^{\infty} W_r x^r = \left( \sum_{r=0}^{\infty} p_r x^r \right) \left( \sum_{r=0}^{\infty} q_r x^r \right).
\]

**Proof.** Since \( (\sum p_r x^r)(\sum q_r x^r) = \sum W_r x^r \), we have

\[
W_r = \sum_{j=0}^{r} p_{r-j} q_j.
\]

From (8) substituting the values of \( W_r \), we have

\[
W_r^2 - W_{r-1} W_{r+1} = \left( \sum_{j=0}^{r} p_{r-j} q_j \right) \left( \sum_{j=0}^{r} p_{r-j} q_j \right)
- \left( \sum_{j=0}^{r+1} p_{r-1-j} q_j \right) \left( \sum_{j=0}^{r+1} p_{r+1-j} q_j \right)
\]

or

\[
W_r^2 - W_{r-1} W_{r+1} = \left( \sum_{j=0}^{r} p_{r-j} q_j \right) \left( \sum_{\lambda=0}^{r} p_{r-\lambda} q_\lambda \right)
- \left( \sum_{j=0}^{r-1} p_{r-1-j} q_j \right) \left( \sum_{\lambda=0}^{r} p_{r+1-\lambda} q_\lambda \right)
+ q_r \sum_{\lambda=0}^{r} p_{r-\lambda} q_\lambda - q_{r+1} \sum_{\lambda=0}^{r-1} p_{r-1-\lambda} q_\lambda.
\]

Now the right side of (9) can be written as I + II + III where

\[
I = \sum_{j=0}^{r-1} \sum_{\lambda=1}^{r} q_j q_\lambda (p_{r-j} p_{r-\lambda} - p_{r-1-j} p_{r+1-\lambda})
\]

\[
II = \sum_{j=0}^{r-1} q_j q_0 (p_{r-j} p_r - p_{r-1-j} p_{r+1})
\]

\[
III = q_r \sum_{\lambda=0}^{r} p_{r-\lambda} q_\lambda - q_{r+1} \sum_{\lambda=0}^{r-1} p_{r-1-\lambda} q_\lambda.
\]

Now III may be rewritten as

\[
p_r q_r + \sum_{\lambda=0}^{r-1} p_\lambda (q_r q_{r-\lambda} - q_{r+1} q_{r-1-\lambda})
\]
and the expression in the parenthesis is nonnegative by the concavity hypothesis. Thus $III \geq 0$. In the same manner it can be proved that $II \geq 0$.

We regard $I$ as a sum of terms arranged in an $r \times r$ matrix $(T_{\lambda})$, with the unusual but understandable indexing $0 \leq j \leq r - 1, 1 \leq \lambda \leq r$:

$$T_{\lambda} = q_{j}\delta_{\lambda}(p_{r-j}p_{r-\lambda} - p_{r-1-j}p_{r+1-\lambda}).$$

The diagonal of this matrix is the set of terms $T_{j, j+1}$, where $0 \leq j \leq r - 1$, and it is clear that all terms on the diagonal vanish. A simple calculation shows that each pair of terms symmetrically positioned with respect to the diagonal has nonnegative sum. Now $I + II + III \geq 0$ and the theorem is established.

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Reference


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