

A CHARACTERIZATION OF TOTALLY REGULAR [$J, f(x)$] TRANSFORMS

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0. **Introduction.** Our main object is to prove a necessary and sufficient condition for [$J, f(x)$] summation methods to be totally regular (§1). But we take this opportunity to establish also an inclusion theorem for regular [$J, f(x)$] means (§2).

A *totally regular* summation method is one which sums a sequence $\{s_n\}$, $n=0, 1, \dots$, to s , whenever $s_n \rightarrow s$, both for s finite and for s infinite. Necessary and sufficient conditions for a triangular method to be totally regular were discovered by W. A. Hurwitz [5] who also characterized the totally regular Hausdorff methods [H, g] as those generated by nondecreasing $g(t)$, $0 \leq t \leq 1$, with $g(0+) = g(0) = 0$ and $g(1) = 1$. Conditions for total regularity of general methods of summation are more complicated than those for a triangular method and were provided by H. Hurwitz [4]. Here we supply a corresponding result for [$J, f(x)$] methods.

The [$J, f(x)$] transform was introduced by Jakimovski [6] in the following way. Let $f(x)$ be a function differentiable infinitely often in $(0, \infty)$. With a sequence $\{s_n\}$, $n=0, 1, \dots$, associate the transform

$$(1) \quad t(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} f^{(n)}(x) s_n, \quad x > 0.$$

The sequence $\{s_n\}$ is said to be summed to s by [$J, f(x)$] if the sum on the right-hand side of (1) exists and if $\lim t(x) = s$, $x \rightarrow \infty$. (If s is infinite we allow this sum to exist in the extended real number system.)

Some well-known methods are included in the [$J, f(x)$] transform. The Borel exponential mean is obtained by taking $f(x) = e^{-x}$, and the Abel scale methods A_γ (see Borwein [2]) are generated by $f(x) = (1+x)^{-\gamma-1}$, $\gamma > -1$, with $\gamma = 0$ giving the Abel method. The Borel transform and the Abel scale methods are totally regular.

It was proved by Jakimovski [6] that the [$J, f(x)$] transform is regular if and only if

$$(2) \quad f(x) = \int_0^\infty e^{-ux} d\alpha(u), \quad x > 0,$$

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where

$$(3) \quad \alpha(u) \in BV[0, \infty), \quad \alpha(0+) = \alpha(0) = 0, \quad \alpha(\infty -) = 1.$$

For the Borel transform $\alpha(u) = 0, 0 \leq u < 1; \alpha(u) = 1, 1 \leq u < \infty$. For the Abel scale

$$\alpha(u) = \frac{1}{\Gamma(\gamma + 1)} \int_0^u t^\gamma e^{-t} dt.$$

I. The characterization. We shall establish the following characterization of totally regular $[J, f(x)]$ methods.

THEOREM 1. *A regular $[J, f(x)]$ transform is totally regular if and only if $\alpha(u)$ is nondecreasing.*

PROOF. Sufficiency is obvious. Conversely, suppose that the $[J, f(x)]$ transform is totally regular; then, by H. Hurwitz [4, Theorem 6],

$$(4) \quad \lim_{x \rightarrow \infty} \sum_{n=0}^{\infty} \left[\left| \frac{x^n}{n!} f^{(n)}(x) \right| - \frac{(-x)^n}{n!} f^{(n)}(x) \right] = 0.$$

Hence for each $v, 0 < v < \infty$, we have

$$(5) \quad \lim_{x \rightarrow \infty} \sum_{n \leq vx} \left[\left| \frac{x^n}{n!} f^{(n)}(x) \right| - \frac{(-x)^n}{n!} f^{(n)}(x) \right] = 0.$$

Since $[J, f(x)]$ is regular, $f(x)$ has the representation (2) where $\alpha(u)$ satisfies (3). We may assume that $\alpha(u)$ is normalized, that is, $\alpha(u) = \frac{1}{2}[\alpha(u+) + \alpha(u-)]$, $0 < u < \infty$. Thus, [7, Theorem 7d, p. 295] for every $t, 0 < t < \infty$,

$$(6) \quad \lim_{x \rightarrow \infty} \sum_{n \leq tx} \frac{(-x)^n}{n!} f^{(n)}(x) = \alpha(t).$$

Define

$$\begin{aligned} \alpha_x(0) &= 0, \\ \alpha_x(t) &= \sum_{n \leq tx} \frac{(-x)^n}{n!} f^{(n)}(x), \quad 0 < t < \infty. \end{aligned}$$

Then the functions $\alpha_x(t), 0 < x < \infty$, are of uniformly bounded variation and, by (6), $\lim_{x \rightarrow \infty} \alpha_x(t) = \alpha(t), 0 \leq t < \infty$. Hence, for $0 < v < \infty$,

$$\begin{aligned}
\int_0^v |d\alpha(t)| &\leq \liminf_{x \rightarrow \infty} \int_0^v |d\alpha_x(t)| \\
&= \liminf_{x \rightarrow \infty} \sum_{n \leq vx} \left| \frac{x^n}{n!} f^{(n)}(x) \right| \\
&\leq \limsup_{x \rightarrow \infty} \sum_{n \leq vx} \left| \frac{x^n}{n!} f^{(n)}(x) \right| \\
&\leq \limsup_{x \rightarrow \infty} \int_0^\infty e^{-ux} \sum_{n \leq vx} \frac{(ux)^n}{n!} |d\alpha(u)|.
\end{aligned}$$

Using again [7, Theorem 7d, p. 295] we see that the last expression equals

$$\int_0^v |d\alpha(u)|.$$

Thus,

$$(7) \quad \lim_{x \rightarrow \infty} \sum_{n \leq vx} \left| \frac{x^n}{n!} f^{(n)}(x) \right| = \int_0^v |d\alpha(t)|.$$

Combining (5) and (7), it follows that

$$\int_0^v |d\alpha(u)| - \alpha(v) = 0, \quad 0 < v < \infty,$$

whence $\alpha(u)$ is nondecreasing. This completes the proof.

2. An inclusion theorem. The following result deals with the relative strength of regular $[J, f(x)]$ methods. It is an extension of Theorem 5.3 of [6].

THEOREM 2. *Suppose that $[J, f(x)]$ is a regular transform and that $\phi(t)$ satisfies (3). Let*

$$g(x) = \int_0^\infty f(xt) d\phi(t), \quad x > 0.$$

If

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} |f^{(n)}(x)| |s_n|$$

exists and is bounded in every finite interval $0 \leq x \leq A$, then $[J, f(x)]$ summability of $\{s_n\}$ to a finite s implies $[J, g(x)]$ summability of $\{s_n\}$ to s . In particular, $[J, g(x)]$ is regular. Moreover, if $\phi(t)$ is nondecreasing the same implication holds also for s infinite.

PROOF. It follows from our assumptions that $g(x)$ is differentiable infinitely often, and that

$$g^{(n)}(x) = \int_0^\infty t^n f^{(n)}(xt) d\phi(t), \quad x > 0.$$

Thus

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} g^{(n)}(x) s_n = \int_0^\infty \left[\sum_{n=0}^{\infty} \frac{(-xt)^n}{n!} f^{(n)}(xt) s_n \right] d\phi(t).$$

Now, if s_n is $[J, f(x)]$ summable to s and if the sum in the brackets above is bounded in every finite interval $0 \leq xt \leq A$, then it follows exactly as for continuous Hausdorff transformations (cf. [3, §11.18]) that s_n is $[J, g(x)]$ summable to s . The rest of the proof is also straightforward.

The following immediate consequence of Theorem 2 is an extension of Theorem 5.4 of [6].

COROLLARY 1. *If the Borel transform of $\{|s_n|\}$ is bounded in every finite interval, and if $\{s_n\}$ is Borel summable to a finite s , then $\{s_n\}$ is summable to s by all regular $[J, f(x)]$ transformations. If s is infinite, then $\{s_n\}$ is summable to s by all totally regular $[J, f(x)]$ transformations.*

This implies in turn

COROLLARY 2. *Ordinary convergence is not equivalent to any $[J, f(x)]$ method.*

PROOF. The sequence $\{0, 1, 0, 1, \dots\}$ is bounded and Borel summable, hence summable by all regular $[J, f(x)]$ methods. But it is divergent.

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