ON SOME SUBSPACES OF BANACH SPACES WHOSE DUALS ARE L_1 SPACES

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1. **Introduction.** Banach spaces whose duals are L_1 spaces were extensively studied in [**3**]. Recently J. Lindenstrauss and D. Wulbert [**4**] proved some results on the classification of these spaces which complemented the results of [**3**]. In [2] the results of [**5**] were applied to prove the following structure theorem for the separable case:

PROPOSITION 1 [**2**, Theorem 2]. *A separable Banach space X satisfies $X^* = L_1$ space if and only if X has a monotone basis $\{x_i\}_{i=1}^{\infty}$ such that for every n the span of $\{x_i\}_{i=1}^{n}$ is isometric to l_∞^n (= the space of n-tuples $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of reals with $\|\lambda\| = \max_{1\le i\le n}|\lambda_i|$).*

We recall that a sequence $\{x_i\}_{i=1}^{\infty}$ in a Banach space X is called a monotone basis of X (see [**1**, p. 67]) if each $x \in X$ has a unique representation $x = \sum_{i=1}^{\infty} \alpha_i x_i$ where $\{\alpha_i\}$ are scalars and the projections P_n on X defined by $P_n(\sum_{i=1}^{\infty} \alpha_i x_i) = \sum_{i=1}^{n} \alpha_i x_i$ are of norm 1.

The purpose of this note is to show that the space c_0 (= the space of real sequences $\rho = \{\rho_n\}_{n=1}^{\infty}$ which converge to 0 with $\|\rho\| = \sup_{1\le n}|\rho_n|$) is the minimal infinite dimensional Banach space whose dual is an L_1 space, namely

THEOREM 1. *Let X be a separable infinite dimensional Banach space whose dual is an L_1 space. Then X has a subspace V isometric to c_0 such that there is a projection of norm 1 from X onto V.*

It is proved in [**3**, p. 67, Corollary 2] that any infinite dimensional Banach space Y with $Y^* = L_1(\mu)$ has a separable infinite dimensional subspace X for which $X^* = L_1(\nu)$. Hence, Theorem 1 implies the following

COROLLARY 1. *Every infinite dimensional Banach space whose dual is an L_1 space contains a subspace isometric to c_0.*

2. **Preliminary lemmas.** Let X be a separable Banach space such that X^* is an L_1 space and assume that $\{x_i\}_{i=1}^{\infty}$ is the monotone basis of X mentioned in Proposition 1. It follows from Proposition 1 that in each subspace $E_n = \operatorname{span}\{x_i\}_{i=1}^{n}$ there is a basis $\{e_i^n\}_{i=1}^{n}$ such that

$$(1) \qquad \left\|\sum_{i=1}^{n} \gamma_i e_i^n\right\| = \max_{1\le i\le n} |\gamma_i|$$

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for any real $\gamma_1, \gamma_2, \cdots, \gamma_n$. It is easy to prove (see [4, §5] and [5]) that the bases $\{e_i^n\}_{i=1}^n$, $n=1, 2, 3, \cdots$ can be chosen such that for every n and $1 \leqq i \leqq n$

$$e_i^n = e_i^{n+1} + a_i^n e_{n+1}^{n+1} \tag{2}$$

where $\sum_{i=1}^n |a_i^n| \leqq 1$. We define now a sequence $\{\phi_i\}_{i=1}^\infty$ of functionals on $\bigcup_{n=1}^\infty E_n$ in the following way: For each $j<n$ and $x=\sum_{i=1}^n b_i e_i^n$, $\phi_j(x)=b_j$. Using (2) it is easy to show that ϕ_j is uniquely defined and linear. Moreover, $|\phi_j(x)| = |b_j| \leqq \max_{1 \leq i \leq n} |b_i| = \|x\|$ and hence, ϕ_j can be extended by continuity to a linear functional of norm 1 on X. Another property of ϕ_j is proved in the following

LEMMA 1. *For each j ϕ_j is an extreme point of the unit ball of X^*.*

PROOF. Assume that $\phi_j = 1/2(f+g)$ where $f, g \in X^*$ and $\|f\| = \|g\| = 1$. For all $n \geqq j$, $1 = f(e_j^n) = g(e_j^n)$ and if, for some $k \neq j$, $f(e_k^n) = \alpha \neq 0$ then

$$f(e_j^n + (\operatorname{sign} \alpha) e_k^n) = 1 + |\alpha| > 1.$$

This contradicts the fact that $\|f\|=1$ (since $\|e_j^n + (\operatorname{sign} \alpha) e_k^n\| = 1$). It follows that $f(e_k^n)=0$ for every $n \geqq j$ and $k \neq j$. Obviously g shares the same property and therefore $f(x)=g(x)=\phi_j(x)$ for any $x \in \bigcup_{n=1}^\infty E_n$. Since $\bigcup_{n=1}^\infty E_n$ is dense in X Lemma 1 is thus proved.

COROLLARY 2. *The closed linear span of $\{\phi_j\}_{j=1}^\infty$ is isometric to l_1.*

PROOF. X^* is an $L_1(\mu)$ space and, as is well known, the extreme points of the unit ball are elements of the form $\phi = \chi_A/\mu(A)$ where A is an atom and χ_A the characteristic function of A. It follows from Lemma 1 that $\|\sum_{i=1}^\infty \alpha_i \phi_i\| = \sum_{i=1}^\infty |\alpha_i|$ for any sequence $\{\alpha_i\}_{i=1}^\infty$ of reals. This concludes the proof.

For every $x = \sum_{j=1}^n b_j e_j^n \in \bigcup_{k=1}^\infty E_k$ we have $\|x\| = \max_{1 \leq j \leq n} |b_j| = \max_{1 \leq j \leq n} |\phi_j(x)| \leqq \sup_{1 \leq j < \infty} |\phi_j(x)| \leqq \|x\|$. It follows that for every $x \in X$

$$\|x\| = \sup_{1 \leqq j < \infty} |\phi_j(x)|. \tag{3}$$

The functionals ϕ_j will play an important role in the sequel.

LEMMA 2. *Let $n(k)$ and $i(k)$ be increasing sequences of positive integers with $i(k) \leqq n(k)$. Then the sequence $\{e_{i(k)}^{n(k)}\}_{k=1}^\infty$ forms a monotone basis in the subspace E which it spans in X. Moreover, in each subspace $F_m = \operatorname{span}\{e_{i(k)}^{n(k)}\}_{k=1}^m$ there is a basis $\{u_i^m\}_{i=1}^m$ satisfying the following*

(a) $u_m^m = e_{i(m)}^{n(m)}$.

(b) *For every m and $i \leqq m$, $u_i^m = u_i^{m+i} + b_i^m u_{m+1}^{m+1}$ where $\sum_{i=1}^m |b_i^m| \leqq 1$.*

(c) $\left\|\sum_{i=1}^{m}\gamma_i u_i^m\right\| = \max_{1\leq i\leq n}|\gamma_i|$ *for any real* $\gamma_1, \cdots, \gamma_m$.

PROOF. Define $u_1^1 = e_{i(1)}^{n(1)}$ and for $k>1$

$$\begin{aligned} u_h^k &= e_{i(k)}^{n(k)} && \text{if } h = k \\ &= \sum_{j=1, j\neq i(k)}^{n(k)} b_{h,j}^k e_j^{n(k)} && \text{if } h < k \quad \text{and} \quad u_h^{k-1} = \sum_{j=1}^{n(k)} b_{h,j}^k e_j^{n(k)}. \end{aligned}$$

The properties (a) and (b) follow directly from the definition of u_j^k and from (c), while the following assertion is an easy consequence of (2): Let $x = \sum_{i=1}^{r} a_i e_i^r = \sum_{i=1}^{s} b_i e_i^s$ where $s>r$. Then $a_i = b_i$ for all $i\leqq r$. (Proof by induction on $s-r$.) In view of this fact and the definition of u_j^k we have the equality $1 = b_{j,i(j)}^k$ for $1\leqq j<k$. Assertion (c) holds for $m=1$ and, assuming its validity for $m=k-1$, we get that $\sum_{h=1}^{k-1}|b_{h,j}^k|\leqq 1$ for all $1\leqq j\leqq n(k)$. (This follows from the fact that $\{e_j^{n(k)}\}_{j=1}^{n(k)}$ is the usual basis in the $l_\infty^{n(k)}$ space $E_{n(k)}$ and that $\left\|\sum_{j=1}^{k-1} \pm u_j^{k-1}\right\| = 1$ for any choice of the signs.) In particular it follows that $b_{p,i(j)}^k = 0$ for all $p\neq j$, $1\leqq p, j\leqq k$. We thus get that

$$\max_{1\leqq j\leqq k-1}|\gamma_j| \leqq \left\|\sum_{j=1}^{k-1}\gamma_j u_j^k\right\| \leqq \left\|\sum_{j=1}^{k-1}\gamma_j u_j^{k-1}\right\| = \max_{1\leqq j\leqq k-1}|\gamma_j|$$

for any real $\gamma_1, \cdots, \gamma_{k-1}$. The definition of u_k^k and the last inequality yield

$$\begin{aligned} \left\|\sum_{j=1}^{k}\gamma_j u_j^k\right\| &= \left\|\sum_{j=1}^{k-1}\gamma_j u_j^k + \gamma_k e_{i(k)}^{n(k)}\right\| \\ &= \max\left\{\left\|\sum_{j=1}^{k-1}\gamma_j u_j^k\right\|, |\gamma_k|\right\} = \max_{1\leqq j\leqq k}|\gamma_j|. \end{aligned} \tag{4}$$

This proves (c).

Now, for any sequence $\{\gamma_j\}_{j=1}^{k+1}$ of reals let $\sum_{j=1}^{k}\gamma_j e_{i(j)}^{n(j)} = \sum_{j=1}^{k}\beta_j u_j^k$. Then by (a), (b) and (c)

$$\begin{aligned} \left\|\sum_{j=1}^{k}\gamma_j e_{i(j)}^{n(j)}\right\| &= \max_{1\leqq j\leqq k}|\beta_j| \\ &\leqq \max\left\{\max_{1\leqq j\leqq k}|\beta_j|, \left|\gamma_{k+1} + \sum_{j=1}^{k}\beta_j b_j^k\right|\right\} \\ &= \left\|\sum_{j=1}^{k}\beta_j u_j^k + \gamma_{k+1}u_{k+1}^{k+1}\right\| \\ &= \left\|\sum_{j=1}^{k+1}\gamma_j u_j^j\right\| = \left\|\sum_{j=1}^{k+1}\gamma_j e_{i(j)}^{n(j)}\right\|. \end{aligned}$$

This inequality shows that $\{e_{i(j)}^{n(j)}\}_{j=1}^{\infty}$ is a monotone basis of E (see [**1**, p. 67]), and Lemma 2 is proved.

By Proposition 1 and Lemma 2 any sequence $\{e_{i(k)}^{n(k)}\}$, where $n(k)$ and $i(k)$ are increasing and $i(k)\leqq n(k)$, span a subspace E whose dual is an L_1 space. As in the definition of the functionals $\{\phi_j\}$ given at the beginning of §2, we define functionals $\{\psi_j\}$ on E by $\psi_j(\sum_{i=1}^{k} b_i u_i^k)=b_j$ if $k\geqq j$ and then extend them by continuity to E. By (3) we have, for every $x\in E$, $\|x\|=\sup_{1\leq j<\infty}|\psi_j(x)|$. Since $u_h^k=\sum_{j=1}^{n(k)} b_{h,j}^k e_j^{n(k)}$ where $b_{h,i(j)}^k=\delta_{h,j}$ we get that $\phi_{i(j)}(u_h^k)=\delta_{h,j}=\psi_j(u_h^k)$ for $1\leqq j,\ h\leqq k$. Hence $\psi_j(x)=\phi_{i(j)}(x)$ for all $x\in E$ and thus

$$\|x\| = \sup_j |\phi_{i(j)}(x)|. \tag{5}$$

In the sequel we will need the following well known

LEMMA 3. *Let $\{w_k\}_{k=1}^{\infty}$ be a sequence of elements of c_0 where $w_k=(\omega_1^k, \omega_2^k, \cdots)$ and assume that the following conditions are satisfied*:

$$\overset{k}{\omega}_k = 1 \qquad \textit{for every } k \tag{6}$$

$$\sum_{k=1;k\neq i}^{\infty} |\overset{k}{\omega}_i| \leqq 1/2. \tag{7}$$

Then $c_0=\operatorname{span}\{w_k\}_{k=1}^{\infty}$.

PROOF. Let U be the operator defined by $U(\sum_{k=1}^{\infty}\gamma_k e_k)=\sum_{k=1}^{\infty}\gamma_k w_k$ where $\{e_k\}_{k=1}^{\infty}$ denotes the usual unit vector basis in c_0. Since U is given by the matrix $|\omega_n^k|$, we get by (6) and (7) that $\|I-U\|\leqq 1/2$, where I is the identity on c_0. It follows that U is invertible and hence $c_0=\operatorname{span}\{w_k\}_{k=1}^{\infty}$.

3. **Proof of Theorem 1.** Let X be a separable Banach space whose dual is an L_1 space. By Proposition 1 there is a system $\{e_i^n\}_{i=1}^{n}$ $n=1, 2, \cdots$ of bases of the type described in the beginning of §2. Consider the functionals $\{\phi_j\}_{j=1}^{\infty}$ constructed there; by the ω^* sequential compactness of the unit ball in X^* there is a subsequence $\{\phi_{n(j)}\}_{j=1}^{\infty}$ of $\{\phi_n\}_{n=1}^{\infty}$ which converges ω^* to a functional ϕ. By Lemma 2, the subspace $E=\operatorname{span}\{e_{n(j)}^{n(j)}\}_{j=1}^{\infty}$ has an L_1 space as a dual and if ψ_j and ψ denote the restrictions of $\phi_{n(j)}$ and ϕ to E respectively, the following conditions are satisfied:

$$\|x\| = \sup_{1\leqq j} |\psi_j(x)|, \tag{8}$$

$$\lim_{j\to\infty} \psi_j(x) = \psi(x), \tag{9}$$

for every $x \in E$. Construct now in E the bases $\{u_j^k\}_{j=1}^k$ $k=1, 2, \cdots$ satisfying (a), (b) and (c) of Lemma 2. It follows from (c) that for any choice of the signs $\|\sum_{i=1}^k \pm u_i^k\| = 1$ and therefore, for any k we have

$$\sum_{i=1}^{k} |\psi(u_i^k)| \leqq 1. \tag{10}$$

Put $p(1)=16$ and choose $q(1)$, $1 \leqq q(1) \leqq 16$, such that $|\psi(u_{q(1)}^{p(1)})| \leqq 16^{-1}$. Denote $\mu_1 = \psi(u_{q(1)}^{p(1)})$ and choose

$$\begin{aligned} p(2) &\geqq 4\cdot 8^2 && \text{if } \mu_1 = 0 \\ &\geqq 32^{-1}\mu_1^{-1} && \text{if } \mu_1 \neq 0 \end{aligned}$$

so large that

$$\left.\begin{aligned} |\psi_h(u_{q(1)}^{p(1)}) - \mu_1| &\leqq 16^{-1} && \text{if } \mu_1 = 0 \\ &\leqq 2^{-1}\mu_1 && \text{if } \mu_1 \neq 0 \end{aligned}\right\} \text{ for all } h \geqq 2^{-1}p(2).$$

By (10) we can choose $q(2)$, $2^{-1}p(2) \leqq q(2) \leqq p(2)$, such that

$$\begin{aligned} |\psi(u_{q(2)}^{p(2)})| &\leqq 2^{-1}\cdot 8^{-2} && \text{if } \mu_1 = 0 \\ &\leqq 2^{-1}\cdot 8^{-1}\mu_1 && \text{if } \mu_1 \neq 0. \end{aligned}$$

Put $\mu_2 = \psi(u_{q(2)}^{p(2)})$. Proceeding by induction we define two increasing sequences $p(k)$ and $q(k)$ of positive integers such that if μ_k denotes $\psi(u_{q(k)}^{p(k)}) = \lim_{j\to\infty} \psi_j(u_{q(k)}^{p(k)})$ then the following inequalities are satisfied:

$$\begin{aligned} |\mu_k| &\leqq 2^{-1}\cdot 8^{-k} && \text{if } \mu_{k-1} = 0 \\ &\leqq 32^{-1}\mu_{k-1} && \text{if } \mu_{k-1} \neq 0 \end{aligned} \tag{11}$$

$$\begin{aligned} |\psi_h(u_{q(k)}^{p(k)}) - \mu_k| &\leqq 2^{-1}\cdot 8^{-k} && \text{if } \mu_{k-1} = 0 \\ &\leqq 2^{-1}\mu_{k-1} && \text{if } \mu_{k-1} \neq 0 \end{aligned} \tag{12}$$

for all $h \geqq 2^{-1}p(k+1)$.

$$p(k+1) \geqq q(k+1) \geqq 2^{-1}p(k+1) > p(k). \tag{13}$$

Now, either (I) there is an N such that $\mu_k \neq 0$ for all $k > N$, or (II) there exists a sequence $\{i(k)\}_{k=1}^\infty$ of positive integers such that $\mu_{i(k)} = 0$ for all k. Suppose that (I) holds, then we define for $k > N$

$$\alpha_k = 1 - \mu_{2k-1}^{-1}\cdot\mu_{2k}\cdot\psi_{q(2k)}(u_{q(2k-1)}^{p(2k-1)})$$

and

$$v_k = \alpha_k^{-1} \cdot (u_{q(2k)}^{p(2k)} - \mu_{2k-1}^{-1} \cdot \mu_{2k} \cdot u_{q(2k-1)}^{p(2k-1)}).$$

Using (11) and (12) one can easily prove that the elements v_k of E satisfy the following conditions:

$$\lim_{j \to \infty} \psi_j(v_k) = 0, \tag{14}$$

$$\psi_{q(2k)}(v_k) = 1, \tag{15}$$

$$\begin{aligned} |\psi_{q(i)}(v_k)| &\leqq 3 \cdot 2^{-1} \cdot 8^{-2k} && \text{if } i > 2k \\ &\leqq 7^{-1} && \text{if } i = 2k-1 \\ &\leqq 0 && \text{if } i < 2k-1. \end{aligned} \tag{16}$$

It follows from (16) that

$$\left.\begin{aligned} \sum_{k=1}^{\infty} |\psi_{q(2n-1)}(v_k)| &\leqq 1/4 \\ \sum_{k=1;\, k \neq n}^{\infty} |\psi_{q(2n)}(v_k)| &\leqq 1/4 \end{aligned}\right\} \quad \text{for } n = 1, 2, \cdots. \tag{17}$$

We apply now (4) and Lemma 2 to the space $E = \operatorname{span}\{u_k^k\}_{k=1}^{\infty}$ where $\{u_i^k\}_{i=1}^k$, ψ_k, $p(k)$ and $q(k)$ play the role of $\{e_i^k\}_{i=1}^k$, ϕ_k, $n(k)$, and $i(k)$ of Lemma 2, respectively, and we get that for each $u \in U = \operatorname{span}\{u_{q(k)}^{p(k)}\}_{k=1}^{\infty}$, $\|u\| = \sup_{1 \leqq k} |\psi_{q(k)}(u)|$. We claim that if $v \in V_0 = \operatorname{span}\{v_k\}_{k=1}^{\infty}$ then $\|v\| = \sup_{1 \leqq k} |\psi_{q(2k)}(v)|$. Indeed, if $v = \sum_{i=1}^n \lambda_i v_i$ and $|\lambda_s| = \max_{1 \leqq i \leqq n} |\lambda_i|$ for some $1 \leqq s \leqq n$ then by (17), for any k, $\psi_{q(2k-1)}(v) \leqq \frac{1}{4}|\lambda_s|$ while $|\psi_{q(2s)}(v)| \geqq 3/4 |\lambda_s|$, and this proves the assertion. Define now a sequence $\{w_k\}_{k=1}^{\infty}$ in c_0 by $w_k = (\psi_{q(2)}(v_k), \psi_{q(4)}(v_k), \psi_{q(6)}(v_k), \cdots)$. Our discussion above shows that the subspace $V_0 = \operatorname{span}\{v_k\}_{k=1}^{\infty}$ of X is isometric to the subspace $W = \operatorname{span}\{w_k\}_{k=1}^{\infty}$ of c_0 (the transformation T: $V_0 \to W$ defined by $T(\sum_1^n \gamma_i v_i) = \sum_{i=1}^n \gamma_i w_i$ for any $\sum_{i=1}^n \gamma_i v_i \in V_0$ can be extended to the desired isometry). The sequence $\{w_k\}_{k=1}^{\infty}$ satisfies, by (15) and (17), the assumptions of Lemma 3 and hence $W = c_0$. It is easy to see that (15), (16), and (17) yield the existence of a sequence $\{y_k\}_{k=1}^{\infty}$ in V_0 such that $\psi_{q(2j)}(y_k) = \delta_{j,k}$, $k, j = 1, 2, \cdots$. Consider the sequence $\{y_{2k-1} - y_{2k}\}_{k=1}^{\infty}$; obviously it spans in V_0 a subspace V isometric to c_0 and

$$\left\| \sum_{k=1}^{\infty} \lambda_k (y_{2k-1} - y_{2k}) \right\| = \max_{1 \leqq k} |\lambda_k| \tag{18}$$

for any sequence $\{\lambda_k\}$ in c_0. We note that ψ_j is defined to be the restriction of $\phi_{n(j)}$ to E and that $\{n(j)\}$ are chosen such that $\lim_{j \to \infty} \phi_{n(j)}(x)$ exists for all $x \in X$. Put $\Phi_j = \phi_{n(q(2j))}$ and define for every $x \in X$

$$Px = 2^{-1}\sum_{j=1}^{\infty}(\bar{\phi}_{2j-1}(x) - \bar{\phi}_{2j}(x))(y_{2j-1} - y_{2j}).$$

The transformation P is obviously a projection from X onto V since $\lim_{j\to\infty}(\phi_{2j-1}(x) - \phi_{2j}(x)) = 0$ for all $x \in X$. Moreover, $\|Px\| = 2^{-1}\sup_{1\leq j}|\bar{\phi}_{2j-1}(x) - \bar{\phi}_{2j}(x)| \leq \|x\|$ by Corollary 2, and hence $\|P\| = 1$.

It remains to consider the possibility (II). If (II) holds, we define $v_k = u_{q(i(k))}^{p(i(k))}$ and $w_k = (\psi_{q(i(1))}(v_k), \psi_{q(i(2))}(v_k), \cdots)$ in c_0. Again, let $W = \operatorname{span}\{w_k\}_{k=1}^{\infty}$ and $V_0 = \operatorname{span}\{v_k\}_{k=1}^{\infty}$, it is easy to show that V_0 and W are isometric and that the sequence $\{w_k\}$ satisfies the assumptions of Lemma 3, and hence V_0 is isometric to c_0. The construction of V and P is done exactly in the same way. This proves Theorem 1.

4. **Concluding remarks.** A. Let X be a separable Banach space whose dual is an L_1 space and assume that the unit cell of X has at least one extreme point. Then X contains a subspace V isometric to c (= the space of convergent sequences $\lambda = \{\lambda_n\}$ of reals with $\|\lambda\| = \sup_n|\lambda_n|$) such that there is a projection of norm 1 from X onto V. This assertion follows from the following considerations (which are only a sketch of the proof): It was observed by Semadeni [6] that a space satisfying the assumptions of our assertion is isometric to a space $\mathfrak{A}(s)$ of affine continuous functions on a Choquet simplex s. Using the methods of [5] one can show that if e denotes the unit function on s then the bases $\{e_i^n\}_{i=1}^{n}$ $n = 1, 2, \cdots$ can be chosen such that $e = e_1^1$ and $\bar{\phi}_j(e) = 1$ for all j. The proof of Theorem 1 now shows that the subspace V spanned by $\{y_k\}_{k=1}^{\infty}$ and e_1^1 in X is isometric to c. The desired projection is given by $P(x) = \phi(x)e_1^1 + \sum_{j=1}^{\infty}(\bar{\phi}_j(x) - \phi(x))y_j$.

B. Given a sequence $\{m(i)\}_{i=1}^{\infty}$ of positive integers, the space X and the system $\{e_i^n\}_{i=1}^{n}$ of bases described at the beginning of §2 we define $x_i = e_i^{s(k)}$ if $s(k-1) < i \leq s(k)$, where $s(0) = 0$ and $s(k) = \sum_{i=1}^{k} m(i)$. The sequence $\{x_i\}_{i=1}^{\infty}$ satisfies the property that

$$\left\|\sum_{i=s(k)+1}^{s(k+1)} \lambda_i x_i\right\| = \max_{s(k) < i \leq s(k+1)} |\lambda_i|$$

and by the proof of Lemma 2 it forms a basis in X. Call the bases $\{x_n\}$ and $\{y_n\}$ *equivalent* if the convergence of $\sum_{i=1}^{\infty}\alpha_i x_i$ is equivalent to the convergence of $\sum_{i=1}^{\infty}\alpha_i y_i$ for any real sequence $\{\alpha_i\}_{i=1}^{\infty}$. It follows that if the basis $\{e_k^k\}_{k=1}^{\infty}$ of X is not equivalent to the unit vector basis of c_0 then by choosing the sequence $\{m(i)\}$ suitably one can construct infinitely many mutually nonequivalent bases of the form $\{x_i\}_{i=1}^{\infty}$.

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