

ELEMENTS OF A NORMED ALGEBRA WHOSE 2^n th POWERS LIE CLOSE TO THE IDENTITY

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In [1], R. H. Cox stated that if x is a square matrix all of whose positive powers lie within a distance $\alpha < 1$ of the identity matrix 1 , then $x = 1$. Nakamura and Yoshida [3] extended this result to bounded operators on Hilbert space; their argument used the mean ergodic theorem. Hirschfeld [2] recently presented a proof, based on spectral theory, showing that the result is valid for elements of any normed algebra. Actually this had been established earlier by Wallen [4], who gave a concise and elementary argument using a significantly weakened hypothesis: he required only that $\|x^n - 1\| = o(n)$ and that $\liminf n^{-1}(\|x - 1\| + \|x^2 - 1\| + \dots + \|x^n - 1\|) < 1$. (Wils [5] also deals with this subject matter; I thank the referee for calling this reference to my attention.)

In this note we examine a different weakening of Cox's hypotheses: we impose conditions only on the 2^n th powers of x .

THEOREM. *Let x be an element of a normed algebra with identity 1 . Let $\alpha_n = \|x^{2^n} - 1\|$. Suppose that $\limsup \alpha_n < 1$. Then for some n , $x^{2^n} = 1$. Assume that in addition every $\alpha_k < 2$. Then $x = 1$.*

PROOF. We write $x^{2^n} = 1 + y_n$. Then $(1 + y_n)^2 = 1 + y_{n+1}$ so that $2y_n = y_{n+1} - y_n^2$, whence

$$(1) \quad 2\alpha_n \leq \alpha_{n+1} + \alpha_n^2.$$

Define $\beta_n = \sup_{k \geq n} \alpha_k$. Then (1) implies

$$2\alpha_n \leq \beta_{n+1} + \beta_n^2.$$

Hence if $k \geq n$ we have

$$2\alpha_k \leq \beta_{k+1} + \beta_k^2 \leq \beta_{n+1} + \beta_n^2.$$

Taking the supremum over all $k \geq n$, it follows that

$$(2) \quad 2\beta_n \leq \beta_{n+1} + \beta_n^2.$$

Because $\beta_{n+1} \leq \beta_n$, this implies

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$$(3) \quad \beta_n \leq \beta_n^2.$$

Now by assumption $\lim \beta_n < 1$. Let n be the smallest integer such that $\beta_n < 1$. Then (3) implies that $\beta_n = 0$, whence $\alpha_n = 0$, i.e. $x^{2^n} = 1$.

Suppose that this n is greater than 0. Then (1), applied to α_{n-1} , says $2\alpha_{n-1} \leq \alpha_{n-1}^2$, i.e. $\alpha_{n-1} \geq 2$. Hence if every $\alpha_k < 2$ we must have $n = 0$ and $x = 1$.

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