

# SUBALGEBRAS OF GROUP ALGEBRAS

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I. Let  $G$  be a locally compact group and  $m$  its Haar measure. For any  $m$ -measurable subset  $S$  of  $G$ , let  $L(S)$  be the subspace of  $L^1(G)$  consisting of elements  $f$  such that  $\int_{G \setminus S} |f| dm = 0$ . If  $S$  is a subsemigroup then  $L(S)$  is a subalgebra of  $L^1(G)$ . Various papers ([4], [5] and [7]) have been devoted to the study of  $L(S)$  and to the question of whether there is a subsemigroup  $T$  such that  $L(S) = L(T)$  whenever  $L(S)$  is an algebra. In [5] it is shown that this is the case whenever  $S$  is contained in a  $\sigma$ -compact subset. A related problem is the following. Let  $dS$  be the set of all  $x$  in  $G$  such that each measurable neighborhood of  $x$  meets  $S$  in a set of positive measure. Whenever  $L(S)$  is an algebra,  $dS$  is a subsemigroup [7], but it need not be true that  $L(S) = L(dS)$ . In this paper we show that in certain cases  $L(S) = L(dS)$ . Using this we give very easy proofs of some of the results in [4] and [7].

Let  $M(G)$  be the Banach  $*$ -algebra of bounded regular Borel measures on  $G$ . (We follow [3] in the definition of Borel subsets etc.) For a Borel subset  $S$  of  $G$ , let  $M(S)$  be the set of  $\mu \in M(G)$  with  $|\mu|(G \setminus S) = 0$ . Suppose that  $S$  is a measurable subsemigroup of  $G$  so that  $L(S)$  is a subalgebra of  $L^1(G)$ . Let  $L(S)^\pi$  be the algebra of left multipliers of  $L(S)$ , i.e. the algebra of bounded linear maps  $\pi$  of  $L(S)$  into itself such that  $\pi(f * g) = \pi f * g$ . Let  $S^\pi = \{x \in G: xS \setminus S \text{ is locally null}\}$ . In this paper we show that if  $G$  is abelian then  $S^\pi$  is closed and that  $M(S^\pi) \subset L(S)^\pi$ , and under certain additional hypotheses  $M(S^\pi) = L(S)^\pi$ . This question has been considered by Birtel in [1] and T. A. Davis in [2]. The question is of some interest since Davis (op. cit.) showed that for abelian  $G$ , the Wiener-Pitt phenomenon occurs for measures in  $L(S)^\pi$ .

II. Let  $C_0(G)$  be the Banach space of continuous complex-valued functions on  $G$  which "vanish at infinity." Let  $f$  be a function on  $G$ ; the support of  $f$ ,  $\text{Supp}(f)$  is the closure of  $\{x \in G: f(x) \neq 0\}$ . Let  $K(G)$  be the subspace of  $C_0(G)$  of functions whose support is compact. Let  $\mu$  be a measure on  $G$ . The support of  $\mu$ ,  $\text{Supp}(\mu)$  is the smallest closed subset of  $G$  whose complement has  $|\mu|$ -measure zero.

It is well known that  $L^1(G)$  may be identified with the subspace of  $M(G)$  of measures that are absolutely continuous with respect to the Haar measure  $m$ , and as such is a two-sided closed ideal in  $M(G)$ . The  $so$ -topology (resp.  $so_r$ -topology) on  $M(G)$  is the coarsest topology

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Received by the editors February 13, 1969.

such that for every  $\lambda$  in  $L^1(G)$  the mapping  $\mu \rightarrow \mu * \lambda$  (resp.  $\mu \rightarrow \lambda * \mu$ ) is a continuous mapping of  $M(G)$  into  $L^1(G)$ .

The first proposition is a well-known property of the support of a measure, and we omit the proof.

**PROPOSITION 1.** *Let  $S$  be a closed subset of  $G$  and let  $\mu \in M(G)$ . Then  $\mu$  is in  $M(S)$  if and only if  $\mu(f) = 0$  for each  $f \in K(G)$  with  $S \cap \text{Supp}(f) = \emptyset$ .*

For  $x \in G$ , let  $\epsilon_x$  be the Dirac measure at  $x$ , i.e.  $\epsilon_x(f) = f(x)$  for  $f \in C_0(G)$ .

**PROPOSITION 2.** *Let  $S$  be a Borel subset of  $G$ .  $S$  is closed if and only if  $M(S)$  is so-closed.*

**PROOF.** Suppose that  $M(S)$  is so-closed and that  $a$  is in the closure of  $S$ . The mapping  $x \mapsto \epsilon_x$  is so-continuous [3, 20.4] so that  $\epsilon_a \in M(S)$ . This means that  $\epsilon_a(G \setminus S) = 0$  and consequently  $a \in S$ . Thus  $S$  is closed.

Now suppose that  $S$  is closed and let  $\mu \in \text{Cl}_{so} M(S)$ . There is a net  $(\mu_j; j \in J)$  in  $M(S)$  such that  $\mu_j \xrightarrow{so} \mu$ , and  $\text{Supp}(\mu_j) \subset S$ . Let  $f$  in  $K(G)$  be such that  $S \cap \text{Supp}(f) = \emptyset$ . There is a symmetric neighborhood  $V$  of  $e$  in  $G$  such that  $S \cap \text{Supp}(f)V = \emptyset$  [3, 4.10]. Let  $\epsilon > 0$  be given, then by the uniform continuity of  $f$  there is a neighborhood  $U$  of  $e$  in  $G$  such that  $|f(xy) - f(x)| \leq \epsilon/2 \|\mu\|$  for all  $x$  in  $G$  and  $y$  in  $U$ . Let  $\lambda$  be a positive measure in  $L^1(G)$  such that  $\text{Supp}(\lambda) \subset U \cap V$ , and  $\|\lambda\| = 1$ . Then  $|\int f(xy) d\lambda(y) - f(x)| \leq \epsilon/2 \|\mu\|$  so that  $|\mu * \lambda(f) - \mu(f)| \leq \epsilon/2$ . If  $y \in V$  and  $x \in G$  is such that  $f(xy) \neq 0$  then  $x \in \text{Supp}(f)V^{-1} \subset \text{Supp}(f)V$ , consequently  $\int_V f(xy) d\lambda(y) = 0$  whenever  $x \in S$ . Thus  $\mu_j * \lambda(f) = 0$ . Since  $\lambda \in L^1(G)$  there is a  $j_0$  such that  $j \geq j_0 \Rightarrow \|\mu_j * \lambda - \mu * \lambda\| \leq \epsilon/2 \|f\|$ . Thus for  $j \geq j_0$ , we have

$$\begin{aligned} |\mu(f)| &\leq |\mu(f) - \mu * \lambda(f)| + |\mu_j * \lambda(f) - \mu * \lambda(f)| + |\mu_j * \lambda(f)| \\ &\leq \epsilon/2 + \|\mu_j * \lambda - \mu * \lambda\| \|f\| \\ &\leq \epsilon. \end{aligned}$$

Therefore we must have  $\mu(f) = 0$  for any  $f \in K(G)$  with  $S \cap \text{Supp}(f) = \emptyset$ . By Proposition 1,  $\mu$  is in  $M(S)$ .

**REMARK.** Let  $\sigma(M(G), C_0(G))$  be the coarsest topology on  $M(G)$  such that for each  $f \in C_0(G)$ , the map  $\mu \mapsto \mu(f)$  is continuous. It is not difficult to show that a Borel subset  $S$  of  $G$  is closed if and only if  $M(S)$  is  $\sigma(M(G), C_0(G))$ -closed.

**III.** Let  $S$  be an  $m$ -measurable subset of  $G$ , define  $dS \subset G$  by  $dS = \{x \in G: \text{if } U \text{ is an open neighborhood of } x, \text{ then } m(U \cap S) > 0\}$ . If  $S$  is locally null then  $L(S) = (0)$ . To avoid this trivial case we shall

suppose throughout that  $S$  is not locally null and then  $dS \neq \emptyset$ . Note that  $dS$  is always closed and that  $L(S) = L(S \cap dS)$  [7].

**THEOREM 1.** *Let  $S$  be an  $m$ -measurable subset of  $G$ . Then  $L(S)$  and  $L(dS)$  are dense in  $M(dS)_{so}$ .*

**PROOF.** Since  $dS$  is closed,  $M(dS)$  is  $so$ -closed (Proposition 2) and since  $L(S) \subset L(dS)$  it suffices to show that  $M(dS) \subset Cl_{so}L(S)$ . Let  $x \in dS, \lambda \in L^1(G)$  and  $\epsilon > 0$  be given. By [3, 20.15] there is a neighborhood  $U$  of  $e$  such that  $\|\mu * \lambda - \lambda\| \leq \epsilon$  for every positive measure  $\mu \in M(G)$  with  $\|\mu\| = 1$  and  $\mu(G \setminus U) = 0$ . Let  $V = xU$ , then  $V$  is a neighborhood of  $x$ . Let  $\mu \in L(S \cap V)$  be a positive measure with  $\|\mu\| = 1$ . Then  $\epsilon_{x^{-1}} * \mu(G \setminus U) = 0$  and  $\|\epsilon_{x^{-1}} * \mu\| = 1$ . Therefore

$$\|\mu * \lambda - \epsilon_x * \lambda\| = \|\epsilon_{x^{-1}} * \mu * \lambda - \lambda\| \leq \epsilon.$$

Clearly  $\mu \in L(S)$  so that  $\epsilon_x \in Cl_{so}L(S)$ . Thus  $x \in dS$  implies  $\epsilon_x \in Cl_{so}L(S)$ . By Proposition 2 of [6], each  $\lambda \in M(dS)$  is an  $so$ -adherence point of the linear span of  $\{\epsilon_x: x \in dS\}$  so it follows that  $M(dS) \subset Cl_{so}L(S)$ .

**PROPOSITION 3.** *Let  $S$  be an  $m$ -measurable subset of  $G$ . If  $L(S)$  is an algebra then  $L(dS) * L(dS) \subset L(S)$ .*

**PROOF.** Let  $\mu \in L(dS)$  and let  $(\mu_j: j \in J) \subset L(S)$  be a net such that  $\mu_j \xrightarrow{so} \mu$ . Let  $\lambda \in L(S)$  then  $\mu_j * \lambda \in L(S)$  and  $\mu_j * \lambda$  converges in the norm to  $\mu * \lambda$ . Therefore  $\mu * \lambda \in L(S)$  since  $L(S)$  is norm closed. This shows that  $L(dS) * L(S) \subset L(S)$ . Theorem 1 remains true if we replace the  $so$ -topology by the  $so_r$ -topology. Consequently for  $\lambda \in L(dS)$  there is a net  $(\lambda_j: j \in J)$  in  $L(S)$  such that  $\lambda_j \xrightarrow{so_r} \lambda$ . For  $\mu \in L(dS)$  we have  $\mu * \lambda_j \in L(S)$  and  $\mu * \lambda_j$  converges to  $\mu * \lambda$  in the norm. This means that  $\mu * \lambda \in L(S)$  which proves the proposition.

**COROLLARY 1.** *If  $L(S)$  is an algebra and if  $e \in dS$ , then  $L(dS) = L(S)$ .*

**PROOF.** This follows from the proposition, since  $e \in dS$  implies that  $L(dS)$  has an approximate unit.

In [7] Simon showed that if  $L(S)$  is a nontrivial  $*$ -subalgebra then  $dS$  is a subgroup of  $G$ . Using this and Corollary 1, we have

**COROLLARY 2.** *Suppose that  $L(S)$  is a  $*$ -subalgebra, then  $L(S) = L(dS)$ .*

**COROLLARY 3.** *Suppose that  $G$  is compact. If  $L(S)$  is a subalgebra of  $L^1(G)$  then  $dS$  is a closed subgroup of  $G$  and  $L(S) = L(dS)$ .*

**PROOF.** A closed subsemigroup of a compact group is necessarily a subgroup.

DEFINITION. Let  $S$  be a measurable subset of  $G$ . If  $L(S)$  is an algebra we shall follow [7] and call  $L(S)$  a vanishing algebra. A vanishing algebra is called a maximal vanishing algebra if it is a proper subalgebra and if for every vanishing algebra  $L(T)$ ,  $L(S) \subset L(T)$  implies  $L(S) = L(T)$  or  $L(T) = L^1(G)$ .

We now give a very easy proof of a result due to Liu [4, Theorem 5].

THEOREM 2. *If  $L(S)$  is a maximal vanishing algebra, then  $dS$  is a closed subsemigroup and  $L(S) = L(dS)$ .*

PROOF. We need only show  $L(S) = L(dS)$ . If  $L(S) \neq L(dS)$  then by the maximality of  $L(S)$ ,  $L(dS) = L^1(G)$ , thus  $d(dS) = dS$  is a subgroup and by Corollary 1,  $L(dS) = L(S)$  a contradiction.

IV. In this section we assume throughout that  $G$  is a locally compact abelian group.

PROPOSITION 4. *Let  $\pi \in L(S)^\pi$ , then given  $x \in dS$ , there is a measure  $\mu \in M(dS)$  such that for each  $f \in L(S)$ ,  $\epsilon_x * \pi f = \mu * f$ .*

PROOF. Let  $(e_j; j \in J)$  be a norm bounded net in  $L(S)$  such that  $e_j \xrightarrow{so} \epsilon_x$ . Then  $(\pi e_j; j \in J)$  is a norm bounded net in  $M(G)$  and there is therefore a subnet  $(\pi e_{j(i)})$  and a  $\mu \in M(G)$  such that  $\pi e_{j(i)} \xrightarrow{so} \mu$  (here  $\sigma$  is the  $\sigma(M(G), C_0(G))$ -topology). Since convolution is  $\sigma(M(G), C_0(G))$ -continuous [6] we have for  $f$  in  $L(S)$ ,  $\pi e_{j(i)} * f \xrightarrow{\sigma} \mu * f$ . On the other hand convolution is also  $so$ -continuous, so that

$$\pi e_{j(i)} * f = \pi(e_{j(i)} * f) = e_{j(i)} * \pi f \xrightarrow{so} \epsilon_x * \pi f.$$

Thus we have  $\epsilon_x * \pi f = \mu * f$ . To complete the proof we need to show that  $\mu \in M(dS)$ , and this follows since  $M(dS)$  is  $\sigma(M(G), C_0(G))$ -closed (remark following Proposition 2) and  $L(S) \subset M(dS)$ .

PROPOSITION 5. *Let  $S$  be a subsemigroup of  $G$  with nonzero Haar measure and let  $S^\pi = \{x \in G: xS \setminus S \text{ is locally null}\}$ . Then  $S^\pi$  is closed.*

PROOF. By Proposition 4,  $L(S)^\pi = \{\mu \in M(G): \mu * f \in L(S) \text{ for all } f \in L(S)\}$  so it follows that  $L(S)^\pi$  is  $so$ -closed since  $L(S)$  is norm closed. It is clear that  $x \in S^\pi$  if and only if  $\epsilon_x \in L(S)^\pi$ . By Proposition 2 of [6]  $x_j \rightarrow x$  if and only if  $\epsilon_{x_j} \rightarrow \epsilon_x$  in the  $so$ -topology. It follows that  $S^\pi$  is closed.

THEOREM 3. *Let  $S$  be a subsemigroup of  $G$  having nonzero Haar measure. Then  $M(S^\pi) \subset L(S)^\pi$  and if  $e \in dS$  then  $M(S^\pi) = M(dS) = L(S)^\pi$ .*

PROOF. Since  $S^\tau$  is closed,  $\mu \in M(S^\tau)$  if and only if  $\text{Supp}(\mu) \subset S^\tau$  so it follows from Proposition 2 of [6] that each  $\mu$  in  $M(S^\tau)$  is an  $so$ -adherence point of the linear span of  $\{\epsilon_x : x \in S^\tau\}$ . Each  $\epsilon_x$  in  $M(S^\tau)$  is in  $L(S)^\tau$  so that  $M(S^\tau) \subset L(S)^\tau$  since both  $M(S^\tau)$  and  $L(S)^\tau$  are  $so$ -closed (Propositions 2 and 5).

If  $e \in dS$  then by Proposition 4,  $L(S)^\tau \subset M(dS)$ . It follows from Proposition 3 that  $M(dS) \subset M(S^\tau)$ .

REMARK. T. A. Davis [2] states that  $L(S)^\tau \subset M(S^\tau)$  (Theorem 3.5) however it appears that his proof is not complete.

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